# A Short Course on Path Integral Methods in the Dynamics of Disordered Spin Systems

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This formalism was introduced into the disordered systems community by de Dominicis and subsequently elaborated and applied by people like Horner, Sommers, Sompolinsky, Zippelius, Crisanti and Rieger in the eighties. It also forms the basis of the more recent work by people like Cugliandolo and Kurchan. Different groups use different names, e.g. 'path integral formalism', 'generating functional formalism', or 'dynamical mean field theory'. Its appeal is twofold. Firstly it allows us to study the dynamics of infinitely large disordered spin system in an *exact* way. Secondly, it provides an intriguing alternative for replica's. There are unfortunately not many texts available for learning about the method, other than research papers. There is just the textbook *Fischer KH, Hertz JA (1991), 'Spin Glasses', Cambridge UP*, which deals only with soft spins (note: Ising systems require a different approach). Like most technical subjects, I believe this is best explained at the work floor level, by actually working out the formalism and carrying out the calculations explicitly in representative models. For these I will take spin systems with SK or Hopfield-type interactions .

## Contents

# 1 Parallel Dynamics Ising SK Model - *The Simplest Case*

## 1.1 Definitions and Properties of Microscopic Dynamics

The parallel SK model is defined as as a collection of $N$ Ising spins $\sigma_i$, with the microscopic state probability $p_t(\boldsymbol{\sigma})$, whose dynamics is given by the following ergodic Markov chain:

$$p_{t+1}(\boldsymbol{\sigma}) = \sum_{\boldsymbol{\sigma}'} W_t\left[\boldsymbol{\sigma};\boldsymbol{\sigma}'\right] p_t(\boldsymbol{\sigma}') \qquad W_t\left[\boldsymbol{\sigma};\boldsymbol{\sigma}'\right] = \prod_i \frac{e^{\beta\sigma_i h_i(\boldsymbol{\sigma}';t)}}{2\cosh[\beta h_i(\boldsymbol{\sigma}';t)]} \tag{1}$$

with the usual local fields $h_i$ and with (symmetric) Gaussian exchange interactions $J_{ij}$:

$$h_i(\boldsymbol{\sigma};t) = \sum_{j\neq i} J_{ij}\sigma_j + \theta_i(t) \qquad J_{ij} = \frac{J_0}{N} + \frac{J}{\sqrt{N}} z_{ij} \tag{2}$$

Here $\langle z_{ij}\rangle = 0$, $\langle z_{ij}^2\rangle = 1$. In the case of stationary external fields, i.e. $\theta_i(t) = \theta_i$, the dynamics (1) obeys detailed balance. The (unique) equilibrium distribution can formally be written in a Boltzmann form (note that here $H$ depends on $\beta$), and one can define a corresponding partition function $Z = \sum_{\boldsymbol{\sigma}} e^{-\beta H(\boldsymbol{\sigma})}$ and a free energy $F = -\frac{1}{\beta}\log Z$:

$$p_{\rm eq}(\boldsymbol{\sigma}) \sim e^{-\beta H(\boldsymbol{\sigma})} \qquad H(\boldsymbol{\sigma}) = -\sum_i \theta_i\sigma_i - \frac{1}{\beta}\sum_i \log 2\cosh[\beta h_i(\boldsymbol{\sigma})] \tag{3}$$

We define correlation and response functions:

$$C_{ij}(t,t') = \langle \sigma_i(t)\sigma_j(t')\rangle, \qquad G_{ij}(t,t') = \frac{\partial}{\partial\theta_j(t')}\langle\sigma_i(t)\rangle \tag{4}$$

In equilibrium one finds time translation invariance, i.e. $C_{ij}(t,t') = C_{ij}(t\text{–}t')$ and $G_{ij}(t,t') = G_{ij}(t\text{–}t')$, and one can prove the following version of the Fluctuation-Dissipation Theorem:

$$G_{ij}(\tau<0) = 0, \qquad G_{ij}(\tau>0) = -\beta[C_{ij}(\tau+1) - C_{ij}(\tau-1)] \tag{5}$$

## 1.2 The Generating Functional

For the process (1), in which we now allow the external fields to be time-dependent, we can write the probability of finding a given path $\boldsymbol{\sigma}(0)\to\boldsymbol{\sigma}(1)\to\ldots\to\boldsymbol{\sigma}(t)$ through phase space as the product of the individual transition probabilities:

$$Prob\left[\boldsymbol{\sigma}(0),\ldots,\boldsymbol{\sigma}(t)\right] = W_{t-1}\left[\boldsymbol{\sigma}(t);\boldsymbol{\sigma}(t\text{–}1)\right]\cdots W_0\left[\boldsymbol{\sigma}(1);\boldsymbol{\sigma}(0)\right] p_0(\boldsymbol{\sigma}(0))$$

(no spin summations). We define a generating functional:

$$Z\left[\boldsymbol{\psi}\right] = \langle e^{-i\sum_{s=1}^t\sum_i \psi_i(s)\sigma_i(s)}\rangle = \sum_{\boldsymbol{\sigma}(0)}\cdots\sum_{\boldsymbol{\sigma}(t)} Prob\left[\boldsymbol{\sigma}(0),\ldots,\boldsymbol{\sigma}(t)\right] e^{-i\sum_s\sum_i\psi_i(s)\sigma_i(s)}$$

It generates the time-dependent spin-averages and correlation- and response functions:

$$\langle\sigma_i(s)\rangle = i\lim_{\boldsymbol{\psi}\to 0}\frac{\partial Z}{\partial\psi_i(s)} \qquad C_{ij}(s,s') = -\lim_{\boldsymbol{\psi}\to 0}\frac{\partial^2 Z}{\partial\psi_i(s)\partial\psi_j(s')} \qquad G_{ij}(s,s') = i\lim_{\boldsymbol{\psi}\to 0}\frac{\partial^2 Z}{\partial\psi_i(s)\partial\theta_j(s')}$$

Upon inserting the transition probabilities (1) we obtain:

$$Z[\boldsymbol{\psi}] = \sum_{\boldsymbol{\sigma}(0)} \cdots \sum_{\boldsymbol{\sigma}(t)} p_0(\boldsymbol{\sigma}(0)) \prod_i \prod_{s=1}^{t} e^{-i\sigma_i(s)[\psi_i(s)+i\beta h_i[\boldsymbol{\sigma}(s-1),s-1]]-\log 2\cosh[\beta h_i[\boldsymbol{\sigma}(s-1);s-1]]}$$

We separate the local fields at any time by inserting appropriate delta-distributions:

$$1 = \int \{d\boldsymbol{h}\} \prod_{s=0}^{t} \prod_i \delta\left[h_i(s) - h_i[\boldsymbol{\sigma}(s);s]\right] = \int \frac{\{d\boldsymbol{h}\}\{d\hat{\boldsymbol{h}}\}}{(2\pi)^{N(t+1)}} \prod_{s=0}^{t} \prod_i e^{i\hat{h}_i(s)\left[h_i(s)-\sum_j J_{ij}\sigma_j(s)-\theta_i(s)\right]}$$

with $\{d\boldsymbol{h}\} = \prod_{i=1}^{N}\prod_{s=0}^{t} dh_i(s)$. Now all spin- and interaction variables appear in exponents:

$$Z[\boldsymbol{\psi}] = \sum_{\boldsymbol{\sigma}(0)} \cdots \sum_{\boldsymbol{\sigma}(t)} p_0(\boldsymbol{\sigma}(0)) \int \frac{\{d\boldsymbol{h}\}\{d\hat{\boldsymbol{h}}\}}{(2\pi)^{N(t+1)}} \prod_{s=1}^{t} e^{i\sum_i \hat{h}_i(s)[h_i(s)-\theta_i(s)]-i\sum_i \sigma_i(s)[\psi_i(s)+i\beta h_i(s-1)]-\sum_i \log 2\cosh[\beta h_i(s-1)]}$$

$$\times e^{i\sum_i \hat{h}_i(0)[h_i(0)-\theta_i(0)]-iJ_0 N\sum_{s=0}^{t}\left[\frac{1}{N}\sum_i \hat{h}_i(s)\right]\left[\frac{1}{N}\sum_j \sigma_j(s)\right]+\frac{iJ_0}{N}\sum_i\sum_{s=0}^{t}\hat{h}_i(s)\sigma_i(s)-\frac{iJ}{\sqrt{N}}\sum_{i\neq j} z_{ij}\sum_{s=0}^{t}\hat{h}_i(s)\sigma_j(s)} \tag{6}$$

with $z_{ij} = z_{ji}$. Averaging of $Z$ over the disorder $\{z_{ij}\}$ (which has become trivial), denoted by $\overline{\cdots}$, gives

$$\overline{\langle\sigma_i(s)\rangle} = i\lim_{\boldsymbol{\psi}\to 0}\frac{\partial\overline{Z}}{\partial\psi_i(s)} \qquad \overline{C}_{ij}(s,s') = -\lim_{\boldsymbol{\psi}\to 0}\frac{\partial^2\overline{Z}}{\partial\psi_i(s)\partial\psi_j(s')} \qquad \overline{G}_{ij}(s,s') = i\lim_{\boldsymbol{\psi}\to 0}\frac{\partial^2\overline{Z}}{\partial\psi_i(s)\partial\theta_j(s')} \tag{7}$$

Overall constants in $\overline{Z}[\boldsymbol{\psi}]$ that do not depend on the external fields $\{\psi_i\}$ can always be recovered *a posteriori*, using the normalisation relation $Z[0] = \overline{Z}[0] = 1$. The latter also allows us to derive additional identities which will play an important role later in the elimination of spurious solutions:

$$0 = \lim_{\boldsymbol{\psi}\to 0}\frac{\partial\overline{Z}[\boldsymbol{\psi}]}{\partial\theta_i(s)} = \lim_{\boldsymbol{\psi}\to 0}\frac{\partial^2\overline{Z}[\boldsymbol{\psi}]}{\partial\theta_i(s)\partial\theta_j(s')} \tag{8}$$

## 1.3 Dynamic Mean Field Theory

The term in (6) containing the disorder $\{z_{ij}\}$ becomes

$$\overline{\left[e^{-\frac{iJ}{\sqrt{N}}\sum_{i\neq j} z_{ij}\sum_{s=0}^{t}\hat{h}_i(s)\sigma_j(s)}\right]} = \overline{\left[e^{-\frac{iJ}{\sqrt{N}}\sum_{i<j} z_{ij}\sum_{s=0}^{t}[\hat{h}_i(s)\sigma_j(s)+\hat{h}_j(s)\sigma_i(s)]}\right]}$$

$$= e^{-\frac{J^2}{4N}\sum_{i\neq j}\left[\sum_{s=0}^{t}[\hat{h}_i(s)\sigma_j(s)+\hat{h}_j(s)\sigma_i(s)]\right]^2}$$

$$= e^{-\frac{1}{2}J^2 N\sum_{s,s'=0}^{t}\left[[\frac{1}{N}\sum_i \hat{h}_i(s)\hat{h}_i(s')][\frac{1}{N}\sum_j \sigma_j(s)\sigma_j(s')]+[\frac{1}{N}\sum_i \hat{h}_i(s)\sigma_i(s')][\frac{1}{N}\sum_j \sigma_j(s)\hat{h}_j(s')]\right]+\mathcal{O}(N^0)}$$

We separate the relevant two-time order parameters by inserting:

$$1 = \left[\frac{N}{2\pi}\right]^{2(t+1)} \int d\boldsymbol{q}\,d\hat{\boldsymbol{q}}\; e^{iN\sum_{s,s'}\hat{q}(s,s')\left[q(s,s')-\frac{1}{N}\sum_i \sigma_i(s)\sigma_i(s')\right]}$$

$$1 = \left[\frac{N}{2\pi}\right]^{2(t+1)} \int d\boldsymbol{Q}\,d\hat{\boldsymbol{Q}}\; e^{iN\sum_{s,s'}\hat{Q}(s,s')\left[Q(s,s')-\frac{1}{N}\sum_i \hat{h}_i(s)\hat{h}_i(s')\right]}$$

$$1 = \left[\frac{N}{2\pi}\right]^{2(t+1)} \int d\boldsymbol{K} d\hat{\boldsymbol{K}} \; e^{iN\sum_{s,s'} \hat{K}(s,s')\left[K(s,s') - \frac{1}{N}\sum_i \sigma_i(s)\hat{h}_i(s')\right]}$$

so that we can write the last term in (6) for finite times $t$ as

$$\int d\boldsymbol{q} d\hat{\boldsymbol{q}} d\boldsymbol{Q} d\hat{\boldsymbol{Q}} d\boldsymbol{K} d\hat{\boldsymbol{K}} \; e^{iN\sum_{s,s'}\left[\hat{q}(s,s')q(s,s') + \hat{Q}(s,s')Q(s,s') + \hat{K}(s,s')K(s,s')\right] - \frac{1}{2}NJ^2\sum_{s,s'}\left[Q(s,s')q(s,s') + K(s,s')K(s',s)\right]}$$

$$\times e^{-i\sum_{s,s'}\sum_i\left[\hat{q}(s,s')\sigma_i(s)\sigma_i(s') + \hat{Q}(s,s')\hat{h}_i(s)\hat{h}_i(s') + \hat{K}(s,s')\sigma_i(s)\hat{h}_i(s')\right] + \mathcal{O}(\log N)} \tag{9}$$

The neglected $\mathcal{O}(\log N)$ terms are functions of the $\boldsymbol{q}, \boldsymbol{Q}$ and $\boldsymbol{K}$; the normalisation $\overline{Z}[0] = 1$ ensures they will drop out of any final result which has the form of an integral dominated by saddle-points.

We now choose the initial state $p_0(\boldsymbol{\sigma}) = \delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}(0)}$. We introduce the auxiliary order parameters $m(s)$ and $k(s)$ via insertion of

$$1 = \left[\frac{N}{2\pi}\right]^{t+1} \int d\boldsymbol{m} d\hat{\boldsymbol{m}} \; e^{iN\sum_s \hat{m}(s)\left[m(s) - \frac{1}{N}\sum_i \sigma_i(s)\right]} \qquad 1 = \left[\frac{N}{2\pi}\right]^{t+1} \int d\boldsymbol{k} d\hat{\boldsymbol{k}} \; e^{iN\sum_s \hat{k}(s)\left[k(s) - \frac{1}{N}\sum_i \hat{h}_i(s)\right]}$$

and insert (9) into (6), which then factorises into single-site contributions.

$$\overline{Z}[\boldsymbol{\psi}] = \int d\boldsymbol{m} d\hat{\boldsymbol{m}} d\boldsymbol{k} d\hat{\boldsymbol{k}} d\boldsymbol{q} d\hat{\boldsymbol{q}} d\boldsymbol{Q} d\hat{\boldsymbol{Q}} d\boldsymbol{K} d\hat{\boldsymbol{K}} \; e^{iN\sum_s\left[\hat{m}(s)m(s) + \hat{k}(s)k(s) - J_0 k(s)m(s)\right] + \mathcal{O}(\log N)}$$

$$\times e^{iN\sum_{s,s'}\left[\hat{q}(s,s')q(s,s') + \hat{Q}(s,s')Q(s,s') + \hat{K}(s,s')K(s,s')\right] - \frac{1}{2}NJ^2\sum_{s,s'}\left[Q(s,s')q(s,s') + K(s,s')K(s',s)\right]}$$

$$\times \prod_i \left\{ \int \{dh\}\{d\hat{h}\} \sum_{\sigma(1),\ldots,\sigma(t)} e^{i\sum_{s\geq 0}\hat{h}(s)[h(s) - \theta_i(s)] - i\sum_{s\geq 1}^{t}\sigma(s)[\psi_i(s) + i\beta h(s-1)] - \sum_{s\geq 0}\log 2\cosh[\beta h(s)]}\right.$$

$$\left. e^{-i\sum_{s,s'\geq 0}\left[\hat{q}(s,s')\sigma(s)\sigma(s') + \hat{Q}(s,s')\hat{h}(s)\hat{h}(s') + \hat{K}(s,s')\sigma(s)\hat{h}(s')\right] - i\sum_{s\geq 0}[\hat{m}(s)\sigma(s) + \hat{k}(s)\hat{h}(s)]} \right\}$$

The generating function is for $N \to \infty$ dominated by a saddle-point. Variation of $\{m(s)\}$ and $\{k(s)\}$ in the extensive exponent gives the saddle-point equations $\hat{m}(s) = J_0 k(s)$ and $\hat{k}(s) = J_0 m(s)$. We can thus simplify the saddle-point problem to

$$\overline{Z}[\boldsymbol{\psi}] = \int d\boldsymbol{m} d\boldsymbol{k} d\boldsymbol{q} d\hat{\boldsymbol{q}} d\boldsymbol{Q} d\hat{\boldsymbol{Q}} d\boldsymbol{K} d\hat{\boldsymbol{K}} \; e^{N\Psi[\boldsymbol{m},\boldsymbol{k};\boldsymbol{q},\hat{\boldsymbol{q}},\boldsymbol{Q},\hat{\boldsymbol{Q}},\boldsymbol{K},\hat{\boldsymbol{K}}] + N\Phi[\boldsymbol{m},\boldsymbol{k};\hat{\boldsymbol{q}},\hat{\boldsymbol{Q}},\hat{\boldsymbol{K}}] + \mathcal{O}(\log N)} \tag{10}$$

with $\Psi$ and $\Phi$ given by:

$$\Psi = iJ_0\sum_{s\geq 0} k(s)m(s) + i\sum_{s,s'\geq 0}\left[\hat{q}(s,s')q(s,s') + \hat{Q}(s,s')Q(s,s') + \hat{K}(s,s')K(s,s')\right]$$

$$-\frac{1}{2}J^2\sum_{s,s'\geq 0}\left[Q(s,s')q(s,s') + K(s,s')K(s',s)\right] \tag{11}$$

$$\Phi = \frac{1}{N}\sum_i \log\left\{ \int \{dh\}\{d\hat{h}\} \sum_{\sigma(1),\ldots,\sigma(t)} e^{i\sum_{s\geq 0}\hat{h}(s)[h(s) - \theta_i(s)] - i\sum_{s\geq 1}^{t}\sigma(s)[\psi_i(s) + i\beta h(s-1)] - \sum_{s\geq 0}\log 2\cosh[\beta h(s)]}\right.$$

$$\left. e^{-i\sum_{s,s'\geq 0}\left[\hat{q}(s,s')\sigma(s)\sigma(s') + \hat{Q}(s,s')\hat{h}(s)\hat{h}(s') + \hat{K}(s,s')\sigma(s)\hat{h}(s')\right] - iJ_0\sum_{s\geq 0}[k(s)\sigma(s) + m(s)\hat{h}(s)]} \right\} \tag{12}$$

We have arrived at a single-spin saddle-point problem, here involving time. The external fields $\{\theta_i(s)\}$ and $\{\psi_i(s)\}$ serve only to identify the physical meaning of our order parameters. Thereafter we can put $\psi_i(s) \to 0$ and $\theta_i(s) \to \theta(s)$ (i.e. a site-independent external field; we will write this as 'sif').

In working out saddle-point equations and derivatives with respect to external fields we will repeatedly encounter the following effective single-spin measure (which will be simplified in due course):

$$\langle f[\{\sigma\},\{h\},\{\hat{h}\}]\rangle_\star = \frac{\int\{dh\}\{d\hat{h}\}\sum_{\sigma(1),\ldots,\sigma(t)} \; M[\{\sigma\},\{h\},\{\hat{h}\}] \; f[\{\sigma\},\{h\},\{\hat{h}\}]}{\int\{dh\}\{d\hat{h}\}\sum_{\sigma(1),\ldots,\sigma(t)} \; M[\{\sigma\},\{h\},\{\hat{h}\}]}$$

with

$$M[\{\sigma\},\{h\},\{\hat{h}\}] = e^{i\sum_{s\geq 0}\hat{h}(s)[h(s)-\theta(s)]+\beta\sum_{s\geq 1}^{t}\sigma(s)h(s-1)-\sum_{s\geq 0}\log 2\cosh[\beta h(s)]}$$
$$\times e^{-i\sum_{s,s'\geq 0}\left[\hat{q}(s,s')\sigma(s)\sigma(s')+\hat{Q}(s,s')\hat{h}(s)\hat{h}(s')+\hat{K}(s,s')\sigma(s)\hat{h}(s')\right]-iJ_0\sum_{s\geq 0}[k(s)\sigma(s)+m(s)\hat{h}(s)]}$$

The order parameters $\{\boldsymbol{m},\boldsymbol{k},\hat{\boldsymbol{q}},\hat{\boldsymbol{Q}},\hat{\boldsymbol{K}}\}$ in the measure $M$ are defined as giving the maximum of the extensive exponent in (10), i.e. they are solved from the extremisation condition $d(\Psi+\Phi)=0$. With this notation, and using $\overline{Z}[0]=1$, we can express field derivatives of the generating function in the following way:

$$\lim_{\boldsymbol{\theta}\to\text{sif}}\lim_{\boldsymbol{\psi}\to 0}\frac{\partial\overline{Z}[\boldsymbol{\psi}]}{\partial\psi_i(s)} = \lim_{\boldsymbol{\theta}\to\text{sif}}\lim_{\boldsymbol{\psi}\to 0}\frac{\int d\boldsymbol{m}\ldots d\hat{\boldsymbol{K}}\; e^{N(\Psi+\Phi)+\mathcal{O}(\log N)}\left[N\frac{\partial\Phi}{\partial\psi_i(s)}\right]}{\int d\boldsymbol{m}\ldots d\hat{\boldsymbol{K}}\; e^{N(\Psi+\Phi)+\mathcal{O}(\log N)}} = -i\langle\sigma(s)\rangle_\star$$

$$\lim_{\boldsymbol{\theta}\to\text{sif}}\lim_{\boldsymbol{\psi}\to 0}\frac{\partial^2\overline{Z}[\boldsymbol{\psi}]}{\partial\psi_i(s)\partial\psi_j(s')} = \lim_{\boldsymbol{\theta}\to\text{sif}}\lim_{\boldsymbol{\psi}\to 0}\frac{\int d\boldsymbol{m}\ldots d\hat{\boldsymbol{K}}\; e^{N(\Psi+\Phi)+\mathcal{O}(\log N)}\left[N\frac{\partial^2\Phi}{\partial\psi_i(s)\partial\psi_j(s')}+N^2\frac{\partial\Phi}{\partial\psi_i(s)}\frac{\partial\Phi}{\partial\psi_j(s')}\right]}{\int d\boldsymbol{m}\ldots d\hat{\boldsymbol{K}}\; e^{N(\Psi+\Phi)+\mathcal{O}(\log N)}}$$
$$= -\delta_{ij}\left[\langle\sigma(s)\sigma(s')\rangle_\star - \langle\sigma(s)\rangle_\star\langle\sigma(s')\rangle_\star\right] - \langle\sigma(s)\rangle_\star\langle\sigma(s')\rangle_\star$$

$$\lim_{\boldsymbol{\theta}\to\text{sif}}\lim_{\boldsymbol{\psi}\to 0}\frac{\partial^2\overline{Z}[\boldsymbol{\psi}]}{\partial\psi_i(s)\partial\theta_j(s')} = \lim_{\boldsymbol{\theta}\to\text{sif}}\lim_{\boldsymbol{\psi}\to 0}\frac{\int d\boldsymbol{m}\ldots d\hat{\boldsymbol{K}}\; e^{N(\Psi+\Phi)+\mathcal{O}(\log N)}\left[N\frac{\partial^2\Phi}{\partial\psi_i(s)\partial\theta_j(s')}+N^2\frac{\partial\Phi}{\partial\psi_i(s)}\frac{\partial\Phi}{\partial\theta_j(s')}\right]}{\int d\boldsymbol{m}\ldots d\hat{\boldsymbol{K}}\; e^{N(\Psi+\Phi)+\mathcal{O}(\log N)}}$$
$$= -\delta_{ij}\left[\langle\sigma(s)\hat{h}(s')\rangle_\star - \langle\sigma(s)\rangle_\star\langle\hat{h}(s')\rangle_\star\right] - \langle\sigma(s)\rangle_\star\langle\hat{h}(s')\rangle_\star$$

$$\lim_{\boldsymbol{\theta}\to\text{sif}}\lim_{\boldsymbol{\psi}\to 0}\frac{\partial\overline{Z}[\boldsymbol{\psi}]}{\partial\theta_i(s)} = \lim_{\boldsymbol{\theta}\to\text{sif}}\lim_{\boldsymbol{\psi}\to 0}\frac{\int d\boldsymbol{m}\ldots d\hat{\boldsymbol{K}}\; e^{N(\Psi+\Phi)+\mathcal{O}(\log N)}\left[N\frac{\partial\Phi}{\partial\theta_i(s)}\right]}{\int d\boldsymbol{m}\ldots d\hat{\boldsymbol{K}}\; e^{N(\Psi+\Phi)+\mathcal{O}(\log N)}} = -i\langle\hat{h}(s)\rangle_\star$$

$$\lim_{\boldsymbol{\theta}\to\text{sif}}\lim_{\boldsymbol{\psi}\to 0}\frac{\partial^2\overline{Z}[\boldsymbol{\psi}]}{\partial\theta_i(s)\partial\theta_j(s')} = \lim_{\boldsymbol{\theta}\to\text{sif}}\lim_{\boldsymbol{\psi}\to 0}\frac{\int d\boldsymbol{m}\ldots d\hat{\boldsymbol{K}}\; e^{N(\Psi+\Phi)+\mathcal{O}(\log N)}\left[N\frac{\partial^2\Phi}{\partial\theta_i(s)\partial\theta_j(s')}+N^2\frac{\partial\Phi}{\partial\theta_i(s)}\frac{\partial\Phi}{\partial\theta_j(s')}\right]}{\int d\boldsymbol{m}\ldots d\hat{\boldsymbol{K}}\; e^{N(\Psi+\Phi)+\mathcal{O}(\log N)}}$$
$$= -\delta_{ij}\left[\langle\hat{h}(s)\hat{h}(s')\rangle_\star - \langle\hat{h}(s)\rangle_\star\langle\hat{h}(s')\rangle_\star\right] - \langle\hat{h}(s)\rangle_\star\langle\hat{h}(s')\rangle_\star$$

Combination of these expressions with equations (7,8) gives the following results for infinitely large systems with site-independent external fields:

$$\langle\hat{h}(s)\rangle_\star = 0 \qquad \langle\hat{h}(s)\hat{h}(s')\rangle_\star = 0 \qquad \overline{\langle\sigma_i(s)\rangle} = \langle\sigma(s)\rangle_\star \tag{13}$$

$$\overline{C}_{ij}(s,s') = \delta_{ij}\langle\sigma(s)\sigma(s')\rangle_\star + (1-\delta_{ij})\langle\sigma(s)\rangle_\star\langle\sigma(s')\rangle_\star \qquad \overline{G}_{ij}(s,s') = -i\delta_{ij}\langle\sigma(s)\hat{h}(s')\rangle_\star \tag{14}$$

We see that the order parameters generated in our theory are essentially single-site ones:

$$C(s,s') = \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\langle\sigma_i(s)\sigma_i(s')\rangle} = \langle\sigma(s)\sigma(s')\rangle_\star \tag{15}$$

$$G(s,s') = \lim_{N\to\infty}\frac{1}{N}\sum_i \overline{\frac{\partial}{\partial\theta_i(s')}\langle\sigma_i(s)\rangle} = -i\langle\sigma(s)\hat{h}(s')\rangle_\star \tag{16}$$

At this stage we work out the saddle-point equations $d(\Psi+\Phi)=0$. These will simplify considerably as a result of (13), and can be expressed in terms of familiar physical quantities due to (15,16). For absent generating fields $\{\psi(s)\}$ and for site-independent external fields $\{\theta(s)\}$ the function $\Phi$ (12) simplifies to

$$\Phi = \log\int\{dh\}\{d\hat{h}\}\sum_{\sigma(1),\ldots,\sigma(t)} M[\{\sigma\},\{h\},\{\hat{h}\}]$$

Variation of $\boldsymbol{m}$ and $\boldsymbol{k}$:

$$\begin{array}{llll}
\frac{\partial}{\partial k(s)}=0: & iJ_0 m(s) + \langle\frac{\partial}{\partial k(s)}\log M\rangle_\star = 0 & \Rightarrow & m(s)=\langle\sigma(s)\rangle_\star \\
\frac{\partial}{\partial m(s)}=0: & iJ_0 k(s) + \langle\frac{\partial}{\partial m(s)}\log M\rangle_\star = 0 & \Rightarrow & k(s)=0
\end{array}$$

Variation of $\{\hat{\boldsymbol{q}},\hat{\boldsymbol{Q}},\hat{\boldsymbol{K}}\}$:

$$\begin{array}{llll}
\frac{\partial}{\partial\hat{q}(s,s')}=0: & iq(s,s') + \langle\frac{\partial}{\partial\hat{q}(s,s')}\log M\rangle_\star = 0 & \Rightarrow & q(s,s')=\langle\sigma(s)\sigma(s')\rangle_\star \\
\frac{\partial}{\partial\hat{Q}(s,s')}=0: & iQ(s,s') + \langle\frac{\partial}{\partial\hat{Q}(s,s')}\log M\rangle_\star = 0 & \Rightarrow & Q(s,s')=0 \\
\frac{\partial}{\partial\hat{K}(s,s')}=0: & iK(s,s') + \langle\frac{\partial}{\partial\hat{K}(s,s')}\log M\rangle_\star = 0 & \Rightarrow & K(s,s')=\langle\sigma(s)\hat{h}(s')\rangle_\star
\end{array}$$

Variation of $\{\boldsymbol{q},\boldsymbol{Q},\boldsymbol{K}\}$:

$$\begin{array}{llll}
\frac{\partial}{\partial q(s,s')}=0: & i\hat{q}(s,s') - \frac{1}{2}J^2 Q(s,s') = 0 & \Rightarrow & \hat{q}(s,s') = -\frac{1}{2}iJ^2Q(s,s')=0 \\
\frac{\partial}{\partial Q(s,s')}=0: & i\hat{Q}(s,s') - \frac{1}{2}J^2 q(s,s') = 0 & \Rightarrow & \hat{Q}(s,s') = -\frac{1}{2}iJ^2q(s,s') \\
\frac{\partial}{\partial K(s,s')}=0: & i\hat{K}(s,s') - J^2 K(s',s) = 0 & \Rightarrow & \hat{K}(s,s') = -iJ^2K(s',s)
\end{array}$$

Note also the validity of the following statement:

$$\frac{\partial}{\partial\theta(s')}\langle\sigma(s)\rangle_\star = -i\langle\sigma(s)\hat{h}(s')\rangle_\star + i\langle\sigma(s)\rangle_\star\langle\hat{h}(s')\rangle_\star = -i\langle\sigma(s)\hat{h}(s')\rangle_\star$$

We can now eliminate all conjugate order parameters. What remains is a dynamical single-spin problem and a set of equations involving only the following three physical objects

$$m(s) = \lim_{N\to\infty}\frac{1}{N}\sum_i\overline{\langle\sigma(s)\rangle} \qquad C(s,s') = \lim_{N\to\infty}\frac{1}{N}\sum_i\overline{\langle\sigma_i(s)\sigma_i(s')\rangle} \qquad G(s,s') = \lim_{N\to\infty}\frac{1}{N}\sum_i\overline{\frac{\partial\langle\sigma_i(s)\rangle}{\partial\theta_i(s')}}$$

Their values are obtained by simultaneously solving the following equations:

$$m(s) = \langle\sigma(s)\rangle_\star \qquad C(s,s') = \langle\sigma(s)\sigma(s')\rangle_\star \qquad G(s,s') = \frac{\partial}{\partial\theta(s')}\langle\sigma(s)\rangle_\star \tag{17}$$

The averages refer to the measure describing our effective single-spin problem. We are only interested in solutions that obey causality, i.e. where $G(s,s')=0$ for $s\le s'$. In fact one can show that the non-causal solutions are automatically ruled out by the temporal boundary conditions. After elimination of the conjugate order parameters via the saddle-point equations, the single spin measure is fully expressed in terms of our three physical order parameters $\{m,C,G\}$. Since we are only interested in expectation values of spins, we can perform the integration over the fields $\{h(s), \hat{h}(s)\}$ in the single-spin measure. The latter becomes:

$$\langle f[\sigma(1),\ldots,\sigma(t)]\rangle_\star = \int dh(0)\ldots dh(t{-}1) \sum_{\sigma(1),\ldots,\sigma(t)} P[h(0),\ldots,h(t{-}1);\sigma(1),\ldots,\sigma(t)] f[\sigma(1),\ldots,\sigma(t)]$$

in which $P[\ldots]$ denotes the probability (density) to find a 'path' $\{h(0),\ldots,h(t-1);\sigma(1),\ldots,\sigma(t)\}$ of spins and fields:

$$P[h(0),\ldots;\sigma(1),\ldots] \sim \int \prod_{s=0}^{t-1} dz(s) \delta\left[h(s) - \theta(s) - J_0 m(s) - J^2 \sum_{s'\geq 0} G(s,s')\sigma(s') - Jz(s)\right] \frac{e^{\beta\sigma(s+1)h(s)}}{2\cosh[\beta h(s)]}$$

$$\times \int \{d\hat{h}\} e^{-\frac{1}{2}J^2 \sum_{s,s'\geq 0} C(s,s')\hat{h}(s)\hat{h}(s') + iJ\sum_{s\geq 0}\hat{h}(s)z(s)}$$

$$\sim \int d\boldsymbol{z}\; e^{-\frac{1}{2}\sum_{s,s'=0}^{t-1} z(s)C^{-1}(s,s')z(s')} \delta\left[h(s) - \theta(s) - J_0 m(s) - J^2 \sum_{s'\geq 0} G(s,s')\sigma(s') - Jz(s)\right] \prod_{s\geq 0} \frac{e^{\beta\sigma(s+1)h(s)}}{2\cosh[\beta h(s)]}$$

These equations are delicate, due to the coupling of fields and spins in the effective measure. They describe a spin exposed to a local field which is composed of

(i) the simple disorder-free mean-field term: $h_{\rm mf}(s) = J_0 m(s) + \theta(s)$

(ii) a retarded self-interaction: $h_{\rm si}(s;\{\sigma\}) = J^2 \sum_{s'<s} G(s,s')\sigma(s')$

(iii) a non-white Gaussian noise: $h_{\rm gn}(s) = Jz(s),\;\; \langle z(s)\rangle = 0,\;\; \langle z(s)z(s')\rangle = C(s,s')$

The spin aligns itself at each time-step stochastically to its local field according to the familiar rule $p_{s+1}(\sigma) = [2\cosh[\beta h(s)]]^{-1} e^{\sigma h(s)}$, with the field $h(s)$ given by the sum of the above three terms. Causality ensures that the following expression for the single-spin measure is properly normalised:

$$\langle f[\sigma(1),\ldots,\sigma(t)]\rangle_\star = \int d\boldsymbol{z}\; P[\boldsymbol{z}] \sum_{\sigma(1),\ldots,\sigma(t)} P[\sigma(1),\ldots,\sigma(t)|\boldsymbol{z}]\; f[\sigma(1),\ldots,\sigma(t)]$$

$$P[\boldsymbol{z}] = \frac{e^{-\frac{1}{2}\sum_{s,s'=0}^{t-1} z(s)C^{-1}(s,s')z(s')}}{\sqrt{(2\pi)^t \mathrm{Det}\boldsymbol{C}}} \qquad P[\sigma(1),\ldots,\sigma(t)|\boldsymbol{z}] = \prod_{s=0}^{t-1} \frac{e^{\beta\sigma(s+1)h(s)}}{2\cosh[\beta h(s)]}\Big|_{h(s)=h_{\rm mf}(s)+h_{\rm si}(s;\{\sigma\})+Jz(s)}$$

Finally we note that the response function can be evaluated somewhat further into an expression where explicit variations of external fields are no longer necessary. The relevant identity to be used is (with $t > t'$):

$$\frac{\partial}{\partial\theta(t')} P[\sigma(1),\ldots,\sigma(t)|\boldsymbol{z}] = \beta\,[\sigma(t'{+}1) - \tanh[\beta h(t')]] \prod_{s=0}^{t-1} \frac{e^{\beta\sigma(s+1)h(s)}}{2\cosh[\beta h(s)]}\Big|_{h(s)=h_{\rm mf}(s)+h_{\rm si}(s;\{\sigma\})+Jz(s)}$$

This enables us to write

$$t > t': \qquad G(t,t') = \frac{\partial}{\partial\theta(t')}\langle\sigma(t)\rangle_\star = \int d\boldsymbol{z}\; P[\boldsymbol{z}] \sum_{\sigma(1),\ldots,\sigma(t)} \sigma(t) \frac{\partial}{\partial\theta(t')} P[\sigma(1),\ldots,\sigma(t)|\boldsymbol{z}]$$

$$= \beta\langle\sigma(t)\,[\sigma(t'{+}1) - \tanh[\beta h(t')]]\rangle_\star \qquad (18)$$

with the usual expression for the effective local field.

### 1.4 The First Few Time-steps

Note that for the first few time-steps one can write explicit expressions for the solutions of our dynamic equations. We will not mention explicitly trivial statements such as $C(s,s)=1$ and $G(s,s'\geq s)=0$. We restrict ourselves to the case of stationary external fields $\theta(s)=\theta$. Our initial state is given by

$$p_0(\sigma)=\frac{1}{2}[1+m(0)]\delta_{\sigma,1}+\frac{1}{2}[1-m(0)]\delta_{\sigma,-1}$$

<u>$t=1$</u>**:**

$$\langle f[\sigma(1)]\rangle_\star=\int dz(0)\ P[z(0)]\sum_{\sigma(1)}P[\sigma(1)|z(0)]\ f[\sigma(1)]$$

$$P[z(0)]=\frac{e^{-\frac{1}{2}z^2(0)/C(0,0)}}{\sqrt{(2\pi)C(0,0)}}\qquad\qquad P[\sigma(1)|z(0)]=\frac{e^{\beta\sigma(1)h(0)}}{2\cosh[\beta h(0)]}\Big|_{h(0)=J_0m(0)+\theta+Jz(0)}$$

Giving:

$$\langle f[\sigma(1)]\rangle_\star=\int Dz\ \sum_{\sigma(1)}\frac{f[\sigma(1)]\ e^{\beta\sigma(1)[J_0m(0)+\theta+Jz]}}{2\cosh[\beta(J_0m(0)+\theta+Jz)]}$$

From which we can calculate the observables

$$m(1)=\int Dz\ \tanh[\beta(J_0m(0)+\theta+Jz)]\qquad\qquad C(1,0)=C(0,1)=m(0)m(1)$$

$$G(1,0)=\beta\left\{1-\int Dz\ \tanh^2[\beta(J_0m(0)+\theta+Jz)]\right\}$$

At the next time-step $t=2$ we will need the inverse and the determinant of the correlation matrix with times $\leq 1$:

$$\boldsymbol{C}^{-1}=\begin{pmatrix}1 & m(0)m(1)\\ m(0)m(1) & 1\end{pmatrix}^{-1}=\frac{1}{1-m^2(0)m^2(1)}\begin{pmatrix}1 & -m(0)m(1)\\ -m(0)m(1) & 1\end{pmatrix}$$

$$\text{Det}\ \boldsymbol{C}=1-m^2(0)m^2(1)$$

<u>$t=2$</u>**:**

$$\langle f[\sigma(1),\sigma(2)]\rangle_\star=\int\frac{dz(0)dz(1)}{\sqrt{(2\pi)^t\text{Det}\boldsymbol{C}}}\ e^{-\frac{1}{2}\sum_{s,s'=0}^{1}z(s)C^{-1}(s,s')z(s')}\sum_{\sigma(1),\sigma(2)}P[\sigma(1),\sigma(2)|z(0),z(1)]\ f[\sigma(1),\sigma(2)]$$

$$P[\sigma(1),\sigma(2)|z(0),z(1)]=\frac{e^{\beta\sigma(1)h(0)+\beta\sigma(2)h(1)}}{4\cosh[\beta h(0)]\cosh[\beta h(1)]}\Big|_{h(s)=h_{\rm mf}(s)+h_{\rm si}(s;\{\sigma\})+Jz(s)}$$

$$\langle f\ [\sigma(1),\sigma(2)]\rangle_*=\int\frac{dz(0)}{\sqrt{2\pi}}\frac{dz(1)}{\sqrt{2\pi}}\frac{1}{\det\mathbf{C}}\ e^{-\frac{1}{2}\sum_{ss'=0}^{1}\mathbf{z}(s')\mathbf{C}^{-1}\mathbf{z}(s)}\sum_{\sigma(1),\sigma(2)}\frac{e^{\beta\sigma(1)h(0)}}{2\cosh\beta h(0)}\frac{e^{\beta\sigma(2)h(1)}}{2\cosh\beta h(1)}\ f\ [\sigma(1),\sigma(2)]$$

where

$$h(0)=J_0m(0)+\theta(0)+Jz(0)\qquad\qquad h(1)=J_0m(1)+\theta(1)+Jz(1)+J^2G(1,0)\sigma(0)$$

$$\mathbf{C}=\begin{pmatrix}1 & m(0)m(1)\\ m(0)m(1) & 1\end{pmatrix}\qquad\qquad \mathbf{C}^{-1}=\frac{1}{1-m^2(0)m^2(1)}\begin{pmatrix}1 & -m(0)m(1)\\ -m(0)m(1) & 1\end{pmatrix}$$

Single-spin average:

$$m(2) \equiv \langle \sigma(2) \rangle_* = \int \frac{dz(0)}{\sqrt{2\pi}} \frac{dz(1)}{\sqrt{2\pi}} \frac{1}{\sqrt{1 - m^2(0)m^2(1)}} \exp\left[ -\frac{1}{2} \sum_{s,s'=0}^{1} \mathbf{z}(s') \mathbf{C}^{-1} \mathbf{z}(s) \right] \times$$

$$\times \tanh \left[ \beta (J_0 m(1) + \theta(1) + J z(1) + J^2 G(1,0) m(0)) \right] =$$

$$\int \mathcal{D}z \, \tanh \beta \left[ J_0 m(1) + \theta(1) + J z + J^2 G(1,0) m(0) \right]$$

Correlation function:

$$C(2,1) \equiv C(1,2) \equiv \langle \sigma(2)\sigma(1) \rangle_* = \int \frac{dz(0)}{\sqrt{2\pi}} \frac{dz(1)}{\sqrt{2\pi}} \frac{1}{\sqrt{1 - m^2(0)m^2(1)}} \exp\left[ -\frac{1}{2} \sum_{s,s'=0}^{1} \mathbf{z}(s') \mathbf{C}^{-1} \mathbf{z}(s) \right] \times$$

$$\times \tanh \beta \left[ J_0 m(0) + \theta(0) + J z(0) \right] \tanh \beta \left[ J_0 m(1) + \theta(1) + J z(1) + J^2 G(1,0) m(0) \right]$$

$$C(2,0) \equiv C(0,2) \equiv \langle \sigma(2)\sigma(1) \rangle_* = m(2) m(0)$$

Response function:

$$G(2,0) \equiv \beta \langle \{ \sigma(2)\sigma(1) - \sigma(2) \tanh \beta \left[ J_0 m(0) + \theta(0) + J m(0) \right] \} \rangle_* = 0$$

$$G(2,1) \equiv \beta \langle \left\{ \sigma^2(2) - \sigma(2) \tanh \beta \left[ J_0 m(1) + \theta(1) + J z(1) + J^2 G(1,0) m(0) \right] \right\} \rangle_* =$$

$$= \beta \left\{ 1 - \int \frac{dz(0)}{\sqrt{2\pi}} \frac{dz(1)}{\sqrt{2\pi}} \frac{1}{\sqrt{1 - m^2(0)m^2(1)}} \exp\left[ -\frac{1}{2} \sum_{s,s'=0}^{1} \mathbf{z}(s') \mathbf{C}^{-1} \mathbf{z}(s) \right] \times \right.$$

$$\left. \tanh^2 \beta \left[ J_0 m(1) + \theta(1) + J z(1) + J^2 G(1,0) m(0) \right] \right\} =$$

$$= \beta \left\{ 1 - \int \mathcal{D}z \tanh^2 \beta \left[ J_0 m(1) + \theta(1) + J z(1) + J^2 G(1,0) m(0) \right] \right\}$$

<u>$t = 3$</u>:

$$\langle f\left[ \sigma(1), \sigma(2), \sigma(3) \right] \rangle_* = \int \frac{dz(0)}{\sqrt{2\pi}} \frac{dz(1)}{\sqrt{2\pi}} \frac{dz(2)}{\sqrt{2\pi}} \frac{1}{\det \mathbf{C}} \exp\left[ -\frac{1}{2} \sum_{s,s'=0}^{2} \mathbf{z}(s') \mathbf{C}^{-1} \mathbf{z}(s) \right] \times$$

$$\times \sum_{\sigma(1),\sigma(2),\sigma(3)} \frac{e^{\beta\sigma(1)h(0)}}{2\cosh \beta h(0)} \frac{e^{\beta\sigma(2)h(1)}}{2\cosh \beta h(1)} \frac{e^{\beta\sigma(3)h(2)}}{2\cosh \beta h(2)} f\left[ \sigma(1), \sigma(2), \sigma(3) \right]$$

where

$$\begin{aligned} h(0) &= J_0 m(0) + \theta(0) + J z(0) \\ h(1) &= J_0 m(1) + \theta(1) + J z(1) + J^2 G(1,0) \sigma(0) \\ h(2) &= J_0 m(2) + \theta(2) + J z(2) + J^2 G(2,0) \sigma(0) + J^2 G(2,1) \sigma(1) \end{aligned}$$

$$\mathbf{C} = \begin{pmatrix} 1 & m(0)m(1) & m(0)m(2) \\ m(0)m(1) & 1 & C(2,1) \\ m(0)m(2) & C(1,2) & 1 \end{pmatrix}$$

$$\mathbf{C}^{-1} = \frac{1}{\det \mathbf{C}} \begin{pmatrix} 1-C^2(1,2) & m(0)[C(2,1)m(2)-m(1)] & m(0)[C(2,1)m(1)-m(2)] \\ m(0)[C(2,1)m(2)-m(1)] & 1-m^2(0)m^2(2) & m^2(0)m(1)m(2)-C(2,1) \\ m(0)[C(2,1)m(1)-m(2)] & m^2(0)m(1)m(2)-C(2,1) & 1-m^2(0)m^2(1) \end{pmatrix}$$

$$\det \mathbf{C} = 1-C^2(2,1)-m^2(0)m^2(1)+2C(2,1)m^2(0)m(1)m(2)-m^2(0)m^2(2)$$

Single-spin average:

$$m(3) \equiv \langle\sigma(3)\rangle_* = \int \frac{dz(0)}{\sqrt{2\pi}}\frac{dz(1)}{\sqrt{2\pi}}\frac{dz(2)}{\sqrt{2\pi}}\frac{1}{\sqrt{\det \mathbf{C}}} \exp\left[-\frac{1}{2}\sum_{s,s'=0}^{2} \mathbf{z}(s')\mathbf{C}^{-1}\mathbf{z}(s)\right] \times$$

$$\times \frac{e^{\beta h(0)} \tanh\beta\left[J_0 m(2)+\theta(2)+Jz(2)+J^2G(2,1)\right] + e^{-\beta h(0)} \tanh\beta\left[J_0 m(2)+\theta(2)+Jz(2)-J^2G(2,1)\right]}{2\cosh\beta h(0)}$$

Correlation function:

$$C(3,0) \equiv C(0,3) \equiv \langle\sigma(3)\sigma(0)\rangle_* = m(3)m(0)$$

$$C(3,1) \equiv C(1,3) \equiv \langle\sigma(3)\sigma(1)\rangle_* = \int \frac{dz(0)}{\sqrt{2\pi}}\frac{dz(1)}{\sqrt{2\pi}}\frac{dz(2)}{\sqrt{2\pi}}\frac{1}{\sqrt{\det \mathbf{C}}} \exp\left[-\frac{1}{2}\sum_{s,s'=0}^{2} \mathbf{z}(s')\mathbf{C}^{-1}\mathbf{z}(s)\right] \times$$

$$\times \frac{e^{\beta h(0)} \tanh\beta\left[J_0 m(2)+\theta(2)+Jz(2)+J^2G(2,1)\right] - e^{-\beta h(0)} \tanh\beta\left[J_0 m(2)+\theta(2)+Jz(2)-J^2G(2,1)\right]}{2\cosh\beta h(0)}$$

$$C(3,2) \equiv C(2,3) \equiv \langle\sigma(3)\sigma(2)\rangle_* = \int \frac{dz(0)}{\sqrt{2\pi}}\frac{dz(1)}{\sqrt{2\pi}}\frac{dz(2)}{\sqrt{2\pi}}\frac{1}{\sqrt{\det \mathbf{C}}} \exp\left[-\frac{1}{2}\sum_{s,s'=0}^{2} \mathbf{z}(s')\mathbf{C}^{-1}\mathbf{z}(s)\right] \times$$

$$\times \frac{\tanh\beta h(1)}{2\cosh\beta h(0)} \left\{e^{\beta h(0)} \tanh\beta\left[J_0 m(2)+\theta(2)+Jz(2)+J^2G(2,1)\right] + \right.$$
$$\left. + e^{-\beta h(0)} \tanh\beta\left[J_0 m(2)+\theta(2)+Jz(2)-J^2G(2,1)\right]\right\}$$

Response function:

$$G(3,0) = \frac{\partial}{\partial\theta(0)} m(3) = \int \frac{dz(0)}{\sqrt{2\pi}}\frac{dz(1)}{\sqrt{2\pi}}\frac{dz(2)}{\sqrt{2\pi}}\frac{1}{\sqrt{\det \mathbf{C}}} \exp\left[-\frac{1}{2}\sum_{s,s'=0}^{2} \mathbf{z}(s')\mathbf{C}^{-1}\mathbf{z}(s)\right] \times$$

$$\times\beta\frac{1}{2\cosh^2\beta h(0)} \frac{\sinh 2\beta J^2 G(2,1)}{\cosh\beta\left[J_0 m(2)+\theta(2)+Jz(2)+J^2G(2,1)\right]\cosh\beta\left[J_0 m(2)+\theta(2)+Jz(2)-J^2G(2,1)\right]}$$

$$G(3,1) = \frac{\partial}{\partial\theta(1)} m(3) = 0$$

$$G(3,2) = \frac{\partial}{\partial\theta(2)} m(3) = \int \frac{dz(0)}{\sqrt{2\pi}}\frac{dz(1)}{\sqrt{2\pi}}\frac{dz(2)}{\sqrt{2\pi}}\frac{1}{\sqrt{\det \mathbf{C}}} \exp\left[-\frac{1}{2}\sum_{s,s'=0}^{2} \mathbf{z}(s')\mathbf{C}^{-1}\mathbf{z}(s)\right] \frac{\beta}{2\cosh\beta h(0)}$$

$$\times\left\{\frac{e^{\beta h(0)}}{\cosh^2\beta\left[J_0 m(2)+\theta(2)+Jz(2)+J^2G(2,1)\right]} + \frac{e^{-\beta h(0)}}{\cosh^2\beta\left[J_0 m(2)+\theta(2)+Jz(2)-J^2G(2,1)\right]}\right\}$$

## 2 Sequential Dynamics Hopfield Model - *Link with Replica Formalism*

### 2.1 Microscopic Dynamics, Technical Subtleties

For sequential dynamics we can either start from a Glauber-type Markov chain description, where at each time-step a randomly drawn spin is updated (and where the duration of each update is defined as $1/N$ so that on $\mathcal{O}(N^0)$ time-scales all spins have been updated once on average) such as

$$p_{t+\frac{1}{N}}(\boldsymbol{\sigma}) = p_t(\boldsymbol{\sigma}) + \frac{1}{N}\sum_{i=1}^{N}\{w_i(F_i\boldsymbol{\sigma})p_t(F_i\boldsymbol{\sigma}) - w_i(\boldsymbol{\sigma})p_t(\boldsymbol{\sigma})\}$$

or we can start from a continuous-time master equation, corresponding to random durations of the sequential update steps (with a Poisson distribution and $1/N$ as the average duration of each update). For $N \to \infty$ the physics of the two starting points can be shown to be identical. However, from a technical point of view, as we will see, the master equation makes our life significantly easier, as I will discuss below.

Thus our microscopic dynamics will now be given by

$$\frac{d}{dt}p_t(\boldsymbol{\sigma}) = \sum_i [p_t(F_i\boldsymbol{\sigma})w_i(F_i\boldsymbol{\sigma};t) - p_t(\boldsymbol{\sigma})w_i(\boldsymbol{\sigma};t)] \tag{19}$$

$$w_i(\boldsymbol{\sigma};t) = \frac{1}{2}\left[1 - \sigma_i \tanh\beta[h_i(\boldsymbol{\sigma};t)]\right] \qquad h_i(\boldsymbol{\sigma};t) = \sum_j J_{ij}\sigma_j + \theta_i(t) \tag{20}$$

We again consider a generating functional, which is now an average over all possible paths $\{\boldsymbol{\sigma}(t)\}$ through phase space:

$$Z[\boldsymbol{\psi}] = \langle e^{-i\sum_i \int_0^t ds\ \psi_i(s)\sigma_i(s)}\rangle \tag{21}$$

Time is now a continuous variable, and ordinary derivatives are replaced by functional derivatives, obeying the usual rules like

$$\frac{\delta}{\delta f(s)}f(t) = \delta[t-s]$$

As with all path integrals, averages such as (21) are understood to be defined in the following way: (*i*) one discretises time in the dynamic equation (19), (*ii*) one calculates the desired average, and subsequently (*iii*) one takes the continuum limit in the resulting expression. The discretised version of our equations, with time-steps of length $\Delta$, would be

$$p_{t+\Delta}(\boldsymbol{\sigma}) = p_t(\boldsymbol{\sigma}) + \Delta\sum_{i=1}^{N}\{w_i(F_i\boldsymbol{\sigma})p_t(F_i\boldsymbol{\sigma}) - w_i(\boldsymbol{\sigma})p_t(\boldsymbol{\sigma})\} \qquad (0 < \Delta \ll 1) \tag{22}$$

$$Z[\boldsymbol{\psi}] = \langle e^{-i\sum_i \sum_{\ell=0}^{L} \Delta\psi_i(\ell.\Delta)\sigma_i(\ell.\Delta)}\rangle \tag{23}$$

From the requirement of self-consistency follows the continuum limit of the Kronecker-$\delta$:

$$\begin{aligned} f(t) &= \textstyle\int dt'\ \delta(t-t')f(t') \\ f(\ell\Delta) &= \textstyle\sum_{\ell'} \Delta\left[\Delta^{-1}\delta_{\ell\ell'}\right] f(\ell'\Delta) \end{aligned}$$

so $\delta_{\ell\ell'} \leftrightarrow \Delta\delta(t-t')$. At the end of our calculation the dependence of any physical observable on $\Delta$, other than via $t = \ell\Delta$, ought to disappear. Note that the 'paths' are not continuous ones in terms of spin states; only time is continuous and the $2^N$ probabilities $p_t(\boldsymbol{\sigma})$ evolve continuously, the spins are still discrete. This poses no problems if we refrain from introducing temporal derivatives of actual spin values (rather than of spin averages).

Let us at this stage return to the discussion of the issue of the alternative starting point of the single-spin updates via a discrete-time Markov chain. At first sight it might seem that (22) is equivalent to starting off with the discrete-time Glauber-type Markov equation, describing single-spin updates at times $t = 1/N, 2/N, 3/N, \ldots$, which after all can be interpreted as a discretised version of the master equation (19), given exactly by expression (22), but with $\Delta = \frac{1}{N}$. There is a subtle, but important difference. In the latter case the parameter that controls the 'discretisation' in (22) (the duration $\Delta$ of the individual iteration steps) induces a coupling of two limits $N \to \infty$ and $\Delta \to 0$ which we would rather control independently. On the other hand, by starting from the master equation (19), the continuum limit $\Delta \to 0$, in which the discretised theory goes to a continuous-time theory, is completely independent of the thermodynamic limit $N \to \infty$. This becomes of vital importance at the point where we wish to use saddle-point integration, since the number of macroscopic observables in this formalism always diverges with the number of discrete time-steps considered. At any time $t$ we will have $\mathcal{O}(t/\Delta)^2$ macroscopic observables to be integrated over in the generating function. Thus, when $\Delta$ is independent of $N$ we can just take the limits $\lim_{\Delta\to 0}\lim_{N\to\infty}$ in precisely that order, so that saddle-point integration is straightforward, but when $\Delta = 1/N$ the integration dimension diverges with $N$, so that straightforward saddle-point integration is forbidden. Physically the two situation would behave similarly, but technically we would in the latter case have to constrain our integrals over order parameters explicitly to build in the condition that in a single iteration step only $\mathcal{O}(N^{-1})$ deviations can be expected, which is a pain.

Averaging the generating functional over the disorder gives the familiar relations (7) for disorder-averaged observables. In order to perform the disorder average, the next step is, as with parallel dynamics, to transport the disorder variables to a more convenient place by inserting appropriate delta-distributions for the local alignment fields:

$$Z[\boldsymbol{\psi}] \sim \langle \int \{d\boldsymbol{h}\}\{d\hat{\boldsymbol{h}}\} \prod_i e^{i\int_0^t ds\ \hat{h}_i(s)[h_i(s)-\theta_i(s)] - i\int_0^t ds\ \sigma_i(s)\psi_i(s)} \prod_{ij} e^{-i\int_0^t ds\ \hat{h}_i(s)J_{ij}\sigma_j(s)} \rangle$$

Here the average $\langle\cdots\rangle$ is still an average over the stochastic process (19). However, due to the $\delta$-distributions introduced for the local alignment fields, this average has become equivalent to the average over a constrained process in which the local fields $\{h(t)\}$ occurring in the transition rates (20) are prescribed for all times. This constrained process can be written as:

$$\frac{d}{dt}\hat{p}_t(\boldsymbol{\sigma}) = \frac{1}{2}\sum_i [1+\sigma_i \tanh[\beta h_i(t)]]\,\hat{p}_t(F_i\boldsymbol{\sigma}) - \frac{1}{2}\sum_i [1-\sigma_i \tanh[\beta h_i(t)]]\,\hat{p}_t(\boldsymbol{\sigma})$$

with the local fields $\{h(t)\}$ precisely as they appear under the integral sign in our path-integral representation of the generating function. Note that it is important that we expect the local field to indeed depend on time in a continuous way for $N \to \infty$, the situation with short-range models would be quite different.

The constrained stochastic process describes independent evolution of the spins. In other words, if we assume that we know the initial microscopic state precisely, $p_0(\boldsymbol{\sigma}) = \delta_{\boldsymbol{\sigma},\boldsymbol{\sigma}(0)}$, it allows for factorised solutions:

$$\hat{p}_t(\boldsymbol{\sigma}) = \prod_i \left[\frac{1}{2}[1+m_i(t)]\,\delta_{\sigma_i,1} + \frac{1}{2}[1-m_i(t)]\,\delta_{\sigma_i,-1}\right] \tag{24}$$

in which the individual site magnetisations $m_i(t) = \langle \sigma_i(t) \rangle$ are the solutions of

$$\frac{d}{dt} m_i(t) = \tanh[\beta h_i(t)] - m_i(t) \qquad m_i(0) = \sigma_i(0)$$

Note that the magnetisation at any site $i$ depends only on $\sigma_i(0)$ and on the fields at that particular site: $m_i(s) = m_i[s; \{h_i(t)\}; \sigma_i(0)]$. This will an important property in order to achieve site-factorization in the generating functional. The generating function can now be written as

$$Z[\boldsymbol{\psi}] \sim \langle \int \{d\boldsymbol{h}\}\{d\hat{\boldsymbol{h}}\} \prod_i e^{i\int_0^t ds\ \hat{h}_i(s)[h_i(s)-\theta_i(s)] - i\int_0^t ds\ \sigma_i(s)\psi_i(s)} \prod_{ij} e^{-i\int_0^t ds\ \hat{h}_i(s)J_{ij}\sigma_j(s)} \rangle_{\text{indep}} \tag{25}$$

Here the brackets $\langle \ldots \rangle_{\text{indep}}$ denote averaging over independent spins according to (24), in which the local fields drive the temporal evolution of the single-site probabilities. For a given site, however, the spin states at different times are not independent. They are coupled through the single-site stochastic equation, which in general gives rise to non-zero auto-correlation functions. In this path-integral representation (25) for the generating function we perform the disorder average.

## 2.2 Dynamic Mean Field Theory

*Derivation of the Saddle-Point Integral.* We now choose for the neural interactions the ones corresponding to the Hopfield model, at first without eliminating self-interactions (elimination of self-interactions translates into just a trivial modification of our final result):

$$J_{ij} = \frac{1}{N}\sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu \qquad \xi_i^\mu \in \{-1, 1\}, \text{ random}$$

We make the condensed ansatz with only one condensed pattern ($\mu = 1$), and denote averaging over the disorder $\{\xi_i^\mu\}$ ($\mu > 1$) by $\overline{\cdots}$.

$$\overline{\prod_{ij} e^{-i\int_0^t ds\ \hat{h}_i(s)J_{ij}\sigma_j(s)}} = e^{-\frac{i}{N}\int_0^t ds [\sum_i \xi_i^1 \hat{h}_i(s)]\left[\sum_j \xi_j^1 \sigma_j(s)\right]} \overline{e^{-\frac{i}{N}\int_0^t ds \sum_{\mu>1}[\sum_i \xi_i^\mu \hat{h}_i(s)]\left[\sum_j \xi_j^\mu \sigma_j(s)\right]}}$$

$$\sim \int d\boldsymbol{m} d\hat{\boldsymbol{m}} d\boldsymbol{k} d\hat{\boldsymbol{k}} e^{iN\int_0^t ds [\hat{m}(s)m(s) + \hat{k}(s)k(s) - k(s)m(s)] - i\sum_s \sum_i [\hat{m}(s)\xi_i^1\sigma_i(s) + \hat{k}(s)\xi_i^1 \hat{h}_i(s)]}$$

$$\times \left[\overline{e^{-\frac{i}{N}\int_0^t ds [\sum_i \xi_i \hat{h}_i(s)]\left[\sum_j \xi_j \sigma_j(s)\right]}}\right]^{p-1} \tag{26}$$

where we inserted delta-distributions to separate the two order parameters

$$m(s) = \frac{1}{N}\sum_i \xi_i^1 \sigma_i(s) \qquad k(s) = \frac{1}{N}\sum_i \xi_i^1 \hat{h}_i(s)$$

The order parameter $m(s)$ is the familiar (condensed) overlap between the system state at time $s$ and pattern $\boldsymbol{\xi}^1$. The disorder average is very similar to the one to be performed in equilibrium calculations. It is again based on linearisation of quadratic exponents with Gaussian integrals, with the notation $D\boldsymbol{x} = \prod_s \left[(2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2(s)} dx(s)\right]$:

$$\overline{e^{-\frac{i}{N}\int_0^t ds [\sum_i \xi_i \hat{h}_i(s)]\left[\sum_j \xi_j \sigma_j(s)\right]}} = \overline{e^{\frac{1}{4i}\int_0^t ds \left\{ \left[\frac{1}{\sqrt{N}}\sum_i \xi_i(\hat{h}_i(s)+\sigma_i(s))\right]^2 - \left[\frac{1}{\sqrt{N}}\sum_i \xi_i(\hat{h}_i(s)-\sigma_i(s))\right]^2 \right\}}}$$

$$= \int \boldsymbol{Dx}\boldsymbol{Dy}\; \overline{e^{\frac{1}{\sqrt{2iN}}\sum_i \xi_i \int_0^t ds\left[x(s)(\hat{h}_i(s)+\sigma_i(s))+iy(s)(\hat{h}_i(s)-\sigma_i(s))\right]}}$$

$$= \int \boldsymbol{Dx}\boldsymbol{Dy} \prod_i \cosh\left[\frac{1}{\sqrt{2iN}}\int_0^t ds\, \left[x(s)(\hat{h}_i(s)+\sigma_i(s))+iy(s)(\hat{h}_i(s)-\sigma_i(s))\right]\right]$$

$$= \int \boldsymbol{Dx}\boldsymbol{Dy} \exp\left\{\frac{1}{4iN}\iint_0^t dsds' \sum_i \left[x(s)(\hat{h}_i(s)+\sigma_i(s))+iy(s)(\hat{h}_i(s)-\sigma_i(s))\right]\right.$$
$$\left.\times \left[x(s')(\hat{h}_i(s')+\sigma_i(s'))+iy(s')(\hat{h}_i(s')-\sigma_i(s'))\right]\right\}$$

We have retained only the leading order in $N$, and neglected $\mathcal{O}(N^{-1})$ contributions. We separate the relevant two-time order parameters:

$$1 = \left[\frac{N}{2\pi}\right]^{2(t/\Delta+1)} \int d\boldsymbol{q}d\hat{\boldsymbol{q}}\; e^{iN\int_0^t dsds'\hat{q}(s,s')\left[q(s,s')-\frac{1}{N}\sum_i \sigma_i(s)\sigma_i(s')\right]}$$

$$1 = \left[\frac{N}{2\pi}\right]^{2(t/\Delta+1)} \int d\boldsymbol{Q}d\hat{\boldsymbol{Q}}\; e^{iN\int_0^t dsds'\hat{Q}(s,s')\left[Q(s,s')-\frac{1}{N}\sum_i \hat{h}_i(s)\hat{h}_i(s')\right]}$$

$$1 = \left[\frac{N}{2\pi}\right]^{2(t/\Delta+1)} \int d\boldsymbol{K}d\hat{\boldsymbol{K}}\; e^{iN\int_0^t dsds'\hat{K}(s,s')\left[K(s,s')-\frac{1}{N}\sum_i \sigma_i(s)\hat{h}_i(s')\right]}$$

so that we can write expression (26) for the last term in (25) in leading order in $N$ as

$$\sim \int d\boldsymbol{m}d\hat{\boldsymbol{m}}d\boldsymbol{k}d\hat{\boldsymbol{k}}d\boldsymbol{q}d\hat{\boldsymbol{q}}d\boldsymbol{Q}d\hat{\boldsymbol{Q}}d\boldsymbol{K}d\hat{\boldsymbol{K}}\; e^{\alpha N\log\Omega[\boldsymbol{q},\boldsymbol{Q},\boldsymbol{K}]+iN\int_0^t dsds'\left[\hat{q}(s,s')q(s,s')+\hat{Q}(s,s')Q(s,s')+\hat{K}(s,s')K(s,s')\right]}$$
$$\times e^{iN\int_0^t ds\left[\hat{m}(s)m(s)+\hat{k}(s)k(s)-k(s)m(s)\right]-i\int_0^t ds\sum_i\left[\hat{m}(s)\xi_i^1\sigma_i(s)+\hat{k}(s)\xi_i^1\hat{h}_i(s)\right]}$$
$$\times e^{-i\int_0^t dsds'\sum_i\left[\hat{q}(s,s')\sigma_i(s)\sigma_i(s')+\hat{Q}(s,s')\hat{h}_i(s)\hat{h}_i(s')+\hat{K}(s,s')\sigma_i(s)\hat{h}_i(s')\right]} \tag{27}$$

with

$$\Omega[\boldsymbol{q},\boldsymbol{Q},\boldsymbol{K}] = \int \boldsymbol{Dx}\boldsymbol{Dy}\; e^{\frac{1}{4i}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}\begin{pmatrix}\boldsymbol{K}+\boldsymbol{K}^\dagger+\boldsymbol{q}+\boldsymbol{Q} & i(\boldsymbol{Q}-\boldsymbol{q}+\boldsymbol{K}-\boldsymbol{K}^\dagger)\\ i(\boldsymbol{Q}-\boldsymbol{q}-\boldsymbol{K}+\boldsymbol{K}^\dagger) & \boldsymbol{K}+\boldsymbol{K}^\dagger-\boldsymbol{q}-\boldsymbol{Q}\end{pmatrix}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}}$$

$$= \exp\left\{-\frac{1}{2}\log\det\left[\mathbf{I}+\frac{1}{2}i\begin{pmatrix}\boldsymbol{K}+\boldsymbol{K}^\dagger+\boldsymbol{q}+\boldsymbol{Q} & i(\boldsymbol{Q}-\boldsymbol{q}+\boldsymbol{K}-\boldsymbol{K}^\dagger)\\ i(\boldsymbol{Q}-\boldsymbol{q}-\boldsymbol{K}+\boldsymbol{K}^\dagger) & \boldsymbol{K}+\boldsymbol{K}^\dagger-\boldsymbol{q}-\boldsymbol{Q}\end{pmatrix}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\log\det\left[\mathbf{I}+\frac{1}{2}i\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\begin{pmatrix}\boldsymbol{Q} & \boldsymbol{K}^\dagger\\ \boldsymbol{K} & \boldsymbol{q}\end{pmatrix}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2}\log\det\left[\frac{1}{2}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\begin{pmatrix}i\boldsymbol{Q} & \mathbf{I}+i\boldsymbol{K}^\dagger\\ \mathbf{I}+i\boldsymbol{K} & i\boldsymbol{q}\end{pmatrix}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix}\right]\right\}$$

so

$$\log\Omega[\boldsymbol{q},\boldsymbol{Q},\boldsymbol{K}] = -\frac{1}{2}\log\det\begin{pmatrix}\boldsymbol{Q} & \boldsymbol{K}^\dagger-i\\ \boldsymbol{K}-i & \boldsymbol{q}\end{pmatrix} \tag{28}$$

For $\Delta\to 0$ the matrices $\boldsymbol{Q}$, $\boldsymbol{K}$ and $\boldsymbol{q}$ become integral operators, the above determinant will be defined as the continuum limit of the determinant that would be obtained upon first discretising time (factors $\Delta$ produced by this procedure will add constants to the exponent in (27), and therefore cannot affect the saddle-point equations). The neglected $\mathcal{O}(N^0)$ contributions are functions of $\boldsymbol{q},\boldsymbol{Q}$ and $\boldsymbol{K}$. Due to the normalisation $\overline{Z}[0]$, they will drop out of any final result which has the form of an integral dominated by saddle-points only.

Insertion of the result (27) into the generating function (25) gives the desired factorisation over sites:

$$\overline{Z}\,[\boldsymbol{\psi}] \sim \int d\boldsymbol{m}d\hat{\boldsymbol{m}}d\boldsymbol{k}d\hat{\boldsymbol{k}}d\boldsymbol{q}d\hat{\boldsymbol{q}}d\boldsymbol{Q}d\hat{\boldsymbol{Q}}d\boldsymbol{K}d\hat{\boldsymbol{K}}\; e^{\alpha N\log\Omega[\boldsymbol{q},\boldsymbol{Q},\boldsymbol{K}]+iN\int_0^t ds\left[\hat{m}(s)m(s)+\hat{k}(s)k(s)-k(s)m(s)\right]}$$

$$e^{iN\int_0^t dsds'\left[\hat{q}(s,s')q(s,s')+\hat{Q}(s,s')Q(s,s')+\hat{K}(s,s')K(s,s')\right]+\mathcal{O}(\log N)}$$

$$\times\prod_i\left\{\int\{dh\}\{d\hat{h}\}e^{i\int_0^t ds\;\hat{h}(s)\left[h(s)-\theta_i(s)-\hat{k}(s)\xi_i^1\right]-i\int_0^t dsds'\;\hat{Q}(s,s')\hat{h}(s)\hat{h}(s')}\right.$$

$$\left.\langle e^{-i\int_0^t ds\;\sigma(s)\left[\psi_i(s)+\hat{m}(s)\xi_i^1\right]-i\int_0^t dsds'\left[\hat{q}(s,s')\sigma(s)\sigma(s')+\hat{K}(s,s')\sigma(s)\hat{h}(s')\right]}\rangle_i\right\}$$

where the spin-average $\langle\ldots\rangle_i$ refers to a single-site one, resulting from the process

$$\frac{d}{dt}\hat{p}_t(\sigma) = \frac{1}{2}\left[1+\sigma\tanh[\beta h(t)]\right]p_t(-\sigma) - \frac{1}{2}\left[1-\sigma\tanh[\beta h(t)]\right]p_t(\sigma)$$

with initial condition $p_0(\sigma) = \delta_{\sigma,\sigma_i(0)}$. The solution of this single-site master equation is

$$\hat{p}_t(\sigma) = \frac{1}{2}\left[1+m_i(t)\right]\delta_{\sigma,1}+\frac{1}{2}\left[1-m_i(t)\right]\delta_{\sigma,-1}$$

$$\frac{d}{dt}m_i(t) = \tanh[\beta h(t)] - m_i(t) \qquad\qquad m_i(0) = \sigma_i(0)$$

At this stage the single-site spin-average depends on the site index $i$ only through the initial condition $m_i(0) = \sigma_i(0)$. The generating function again acquires the form of an integral which for $N\to\infty$ will be dominated by a saddle-point. Variation of the quantities $\{m(s)\}$ and $\{k(s)\}$ in the extensive exponent gives the saddle-point equations $\hat{m}(s) = k(s)$ and $\hat{k}(s) = m(s)$, so that we can simplify the saddle-point problem to

$$Z\,[\boldsymbol{\psi}] \sim \int d\boldsymbol{m}d\boldsymbol{k}d\boldsymbol{q}d\hat{\boldsymbol{q}}d\boldsymbol{Q}d\hat{\boldsymbol{Q}}d\boldsymbol{K}d\hat{\boldsymbol{K}}\; e^{N\left[\alpha\log\Omega[\boldsymbol{q},\boldsymbol{Q},\boldsymbol{K}]+\Psi[\boldsymbol{m},\boldsymbol{k};\boldsymbol{q},\boldsymbol{Q},\boldsymbol{K};\hat{\boldsymbol{q}},\hat{\boldsymbol{Q}},\hat{\boldsymbol{K}}]+\Phi[\boldsymbol{m},\boldsymbol{k};\hat{\boldsymbol{q}},\hat{\boldsymbol{Q}},\hat{\boldsymbol{K}}]\right]+\mathcal{O}(\log N)} \tag{29}$$

with $\Omega$ as given by (28) and with $\Psi$ and $\Phi$ given by:

$$\Psi = i\int_0^t dsds'\left[\hat{q}(s,s')q(s,s')+\hat{Q}(s,s')Q(s,s')+\hat{K}(s,s')K(s,s')\right]+i\int_0^t ds\;k(s)m(s) \tag{30}$$

$$\Phi = \frac{1}{N}\sum_i\log\left\{\int\{dh\}\{d\hat{h}\}e^{-i\int_0^t dsds'\;\hat{Q}(s,s')\hat{h}(s)\hat{h}(s')+i\int_0^t ds\;\hat{h}(s)\left[h(s)-\theta_i(s)-m(s)\xi_i^1\right]}\right.$$
$$\left.\langle e^{-i\int_0^t ds\;\sigma(s)\left[\psi_i(s)+k(s)\xi_i^1\right]-i\int_0^t dsds'\left[\hat{q}(s,s')\sigma(s)\sigma(s')+\hat{K}(s,s')\sigma(s)\hat{h}(s')\right]}\rangle_i\right\} \tag{31}$$

*Physical Meaning of the Order Parameters and Simplification of the Saddle-Point Problem due to Normalisation and/or Causality.* Using $Z[0] = 1$, and the property that the integral (29) evaluated for $\boldsymbol{\psi}\to 0$ gives just a constant, we obtain from (29) the relevant observables. These observables are expressed in terms of the order parameters introduced, upon deriving saddle-point equations by variation of $\{k(s)\}$, $\{\hat{q}(s,s')\}$ and $\{\hat{K}(s,s')\}$:

$$m(s)|_{\psi=0,\ \text{saddle}} = \frac{1}{N}\sum_i \xi_i^1 \overline{\langle\sigma_i(s)\rangle} \tag{32}$$

$$q(s,s')|_{\psi=0,\ \text{saddle}} = C(s,s') = \frac{1}{N}\sum_i \overline{\langle\sigma_i(s)\sigma_i(s')\rangle} \tag{33}$$

$$K(s,s')|_{\psi=0,\ \text{saddle}} = iG(s,s') = \frac{i}{N}\sum_i \frac{\partial\overline{\langle\sigma_i(s)\rangle}}{\partial\theta_i(s')} \tag{34}$$

Causality and the normalisation conditions (8), applied to (29), give the by now familiar simplifications:

$$s \leq s': \qquad K(s,s')|_{\psi=0,\ \text{saddle}} = 0 \tag{35}$$

$$0 = \frac{1}{N}\sum_i \xi_i^1 \frac{1}{\overline{Z}[0]}\frac{\partial\overline{Z}[0]}{\partial\theta_i(s)} = \frac{\partial\Phi}{\partial m(s)}|_{\psi=0,\ \text{saddle}}$$

$$0 = \frac{1}{N}\sum_i \frac{1}{\overline{Z}[0]}\frac{\partial^2\overline{Z}[0]}{\partial\theta_i(s)\partial\theta_i(s')} = -i\frac{\partial\Phi}{\partial\hat{Q}(s,s')}|_{\psi=0,\ \text{saddle}}$$

which in combination with the saddle-point equations for $m(s)$ and $\hat{Q}(s,s')$ lead to:

$$k(s)|_{\psi=0,\ \text{saddle}} = Q(s,s')|_{\psi=0,\ \text{saddle}} = 0 \tag{36}$$

Near the relevant saddle-point, where $Q(s,s') = 0$ and $K(s,s') = 0$ $(s' \geq s)$, we can calculate the derivatives of $\log\Omega$:

$$\begin{aligned}
\delta\log\Omega &= -\frac{1}{2}\log\det\left[\begin{pmatrix} 0 & \boldsymbol{K}^\dagger - i \\ \boldsymbol{K}-i & \boldsymbol{q}\end{pmatrix} + \begin{pmatrix}\delta\boldsymbol{Q} & \delta\boldsymbol{K}^\dagger \\ \delta\boldsymbol{K} & \delta\boldsymbol{q}\end{pmatrix}\right] + \frac{1}{2}\log\det\begin{pmatrix} 0 & \boldsymbol{K}^\dagger - i \\ \boldsymbol{K}-i & \boldsymbol{q}\end{pmatrix} \\
&= -\frac{1}{2}\log\det\left[\mathbf{I} + \begin{pmatrix} 0 & \boldsymbol{K}^\dagger - i \\ \boldsymbol{K}-i & \boldsymbol{q}\end{pmatrix}^{-1}\begin{pmatrix}\delta\boldsymbol{Q} & \delta\boldsymbol{K}^\dagger \\ \delta\boldsymbol{K} & \delta\boldsymbol{q}\end{pmatrix}\right] \\
&= -\frac{1}{2}\mathrm{Tr}\ \log\left[\mathbf{I} + \begin{pmatrix} 0 & \boldsymbol{K}^\dagger - i \\ \boldsymbol{K}-i & \boldsymbol{q}\end{pmatrix}^{-1}\begin{pmatrix}\delta\boldsymbol{Q} & \delta\boldsymbol{K}^\dagger \\ \delta\boldsymbol{K} & \delta\boldsymbol{q}\end{pmatrix}\right] \\
&= -\frac{1}{2}\mathrm{Tr}\left[\begin{pmatrix} 0 & \boldsymbol{K}^\dagger - i \\ \boldsymbol{K}-i & \boldsymbol{q}\end{pmatrix}^{-1}\begin{pmatrix}\delta\boldsymbol{Q} & \delta\boldsymbol{K}^\dagger \\ \delta\boldsymbol{K} & \delta\boldsymbol{q}\end{pmatrix}\right] + \ldots
\end{aligned}$$

In the relevant saddle-point itself we find

$$\begin{pmatrix} 0 & \boldsymbol{K}^\dagger - i \\ \boldsymbol{K}-i & \boldsymbol{q}\end{pmatrix}^{-1} = \begin{pmatrix} -(\boldsymbol{K}-i)^{-1}\boldsymbol{q}(\boldsymbol{K}^\dagger-i)^{-1} & (\boldsymbol{K}-i)^{-1} \\ (\boldsymbol{K}^\dagger-i)^{-1} & 0\end{pmatrix}$$

and the variation of $\log\Omega$ around the physical saddle-point simply becomes

$$\delta\log\Omega = \frac{1}{2}\mathrm{Tr}\left[(\boldsymbol{K}-i\mathbf{I})^{-1}\boldsymbol{q}(\boldsymbol{K}^\dagger-i\mathbf{I})^{-1}\delta\boldsymbol{Q}\right] - \mathrm{Tr}\left[(\boldsymbol{K}-i\mathbf{I})^{-1}\delta\boldsymbol{K}\right] \tag{37}$$

in which operator products are defined as $(\boldsymbol{AB})(t,t')=\int dt'' A(t,t'')B(t'',t')$, the unity operator is defined as $\mathbf{I}(t,t')=\delta(t-t')$, and with the trace now defined as an integral:

$$\mathrm{Tr}\boldsymbol{A}=\Delta^{-1}\int_0^t ds\ A(s,s)$$

The external fields are no longer needed and can be put to zero. The $\psi_i(s)=\theta_i(s)=0$ saddle-point equations are obtained by straightforward differentiation of the exponent in (29), and differ only from the corresponding parallel dynamics ones in the actual meaning of the single-site spin-averages and in the continuous time. The order parameters $q(s,s')$ and $K(s,s')$ are replaced by correlation- and response functions with (33,34). Derivatives of $\log\Omega$ at the relevant saddle-point generate the expressions for the conjugate order parameters:

$$\hat{\boldsymbol{Q}}=-\frac{1}{2}\alpha i(\boldsymbol{G}-\mathbf{I})^{-1}\boldsymbol{C}(\boldsymbol{G}^\dagger-\mathbf{I})^{-1}\qquad \hat{\boldsymbol{q}}=0\qquad \hat{\boldsymbol{K}}=-\alpha(\boldsymbol{G}^\dagger-\mathbf{I})^{-1}$$

The final saddle-point equations contain only the condensed overlap and the correlation- and response functions:

$$m(s)=\frac{1}{N}\sum_i \xi_i^1\frac{\langle\!\langle\sigma(s)\rangle\!\rangle_i}{\langle\!\langle 1\rangle\!\rangle_i}\qquad C(s,s')=\frac{1}{N}\sum_i\frac{\langle\!\langle\sigma(s)\sigma(s')\rangle\!\rangle_i}{\langle\!\langle 1\rangle\!\rangle_i}\qquad G(s,s')=-\frac{i}{N}\sum_i\frac{\langle\!\langle\sigma(s)\hat{h}(s')\rangle\!\rangle_i}{\langle\!\langle 1\rangle\!\rangle_i}$$

In which the short-hand $\langle\!\langle f(\boldsymbol{\sigma};\hat{\boldsymbol{h}})\rangle\!\rangle_i$ stands for

$$\langle\!\langle f(\boldsymbol{\sigma};\hat{\boldsymbol{h}})\rangle\!\rangle_i=\int\{dh\}\{d\hat{h}\}\ e^{-\frac{1}{2}\alpha\int_0^t dsds'\ \hat{h}(s)\left[(\boldsymbol{G}-\mathbf{I})^{-1}\boldsymbol{C}(\boldsymbol{G}^\dagger-\mathbf{I})^{-1}\right](s,s')\hat{h}(s')+i\int_0^t ds\ \hat{h}(s)\left[h(s)-m(s)\xi_i^1\right]}$$

$$\times\langle f(\boldsymbol{\sigma};\hat{\boldsymbol{h}})e^{i\alpha\int_0^t dsds'\ \sigma(s)[\boldsymbol{G}^\dagger-\mathbf{I}]^{-1}(s,s')\hat{h}(s')}\rangle_i$$

We eliminate the remaining site indices $i$ with the gauge transformation $\hat{\boldsymbol{h}}\to\xi_i^1\hat{\boldsymbol{h}}$, $\boldsymbol{h}\to\xi_i^1\boldsymbol{h}$, $\sigma(s)\to\xi_i^1\sigma(s)$ (the local re-definition of 'up' and 'down' which turns the overlaps $m(s)$ into magnetisations). The result is:

$$m(s)=\frac{\langle\!\langle\sigma(s)\rangle\!\rangle}{\langle\!\langle 1\rangle\!\rangle}\qquad C(s,s')=\frac{\langle\!\langle\sigma(s)\sigma(s')\rangle\!\rangle}{\langle\!\langle 1\rangle\!\rangle}\qquad G(s,s')=-i\frac{\langle\!\langle\sigma(s)\hat{h}(s')\rangle\!\rangle}{\langle\!\langle 1\rangle\!\rangle}\tag{38}$$

in which now

$$\langle\!\langle f(\boldsymbol{\sigma};\hat{\boldsymbol{h}})\rangle\!\rangle=\int\{dh\}\{d\hat{h}\}\ e^{-\frac{1}{2}\alpha\int_0^t dsds'\ \hat{h}(s)\left[(\boldsymbol{G}-\mathbf{I})^{-1}\boldsymbol{C}(\boldsymbol{G}^\dagger-\mathbf{I})^{-1}\right](s,s')\hat{h}(s')}$$

$$e^{i\int_0^t ds\ \hat{h}(s)[h(s)-m(s)]}\langle f(\boldsymbol{\sigma};\hat{\boldsymbol{h}})e^{i\alpha\int_0^t dsds'\ \sigma(s)[\boldsymbol{G}^\dagger-\mathbf{I}]^{-1}(s,s')\hat{h}(s')}\rangle\tag{39}$$

with the spin average resulting from the gauge-transformed single-site process

$$\frac{d}{dt}\hat{p}_t(\sigma)=\frac{1}{2}\left[1+\sigma\tanh[\beta h(t)]\right]\hat{p}_t(-\sigma)-\frac{1}{2}\left[1-\sigma\tanh[\beta h(t)]\right]\hat{p}_t(\sigma)\tag{40}$$

$$\hat{p}_0(\sigma)=\frac{1}{2}[1+m(0)]\delta_{\sigma,1}+\frac{1}{2}[1-m(0)]\delta_{\sigma,-1}$$

In particular one obtains

$$\frac{d}{dt}\langle\!\langle\sigma(t)\rangle\!\rangle=\langle\!\langle\tanh[\beta h(t)]\rangle\!\rangle-\langle\!\langle\sigma(t)\rangle\!\rangle\tag{41}$$

$$\tau > 0: \qquad \frac{d}{d\tau}\langle\langle\sigma(t+\tau)\sigma(t)\rangle\rangle = \langle\langle\Big[\tanh[\beta h(t+\tau)] - \sigma(t+\tau)\Big]\,\sigma(t)\rangle\rangle \tag{42}$$

$$\tau > 0: \qquad \frac{d}{d\tau}\langle\langle\sigma(t+\tau)\tanh[\beta h(t)]\rangle\rangle = \langle\langle\Big[\tanh[\beta h(t+\tau)] - \sigma(t+\tau)\Big]\tanh[\beta h(t)]\rangle\rangle \tag{43}$$

We now proceed again by performing the integrals over the conjugate fields $\hat{\boldsymbol{h}}$, and we substitute for the effective local fields $h(s) \to h(\boldsymbol{\sigma}, \boldsymbol{\phi}; s)$, with

$$h(\boldsymbol{\sigma}, \boldsymbol{\phi}; s) = m(s) + \sqrt{\alpha}\phi(s) + \alpha\int_0^s ds'(\mathbf{1}-\boldsymbol{G})^{-1}(s,s')\sigma(s') \tag{44}$$

Insertion into (39) gives us the following

$$\langle\langle f(\boldsymbol{\sigma}; \hat{\boldsymbol{h}})\rangle\rangle \sim \int\{dh\}\{d\phi\}\ e^{-\frac{1}{2}\alpha\int_0^t dsds'\ \hat{h}(s)\left[(\boldsymbol{G}-\mathbf{1})^{-1}\boldsymbol{C}(\boldsymbol{G}^\dagger-\mathbf{1})^{-1}\right](s,s')\hat{h}(s')+i\sqrt{\alpha}\int_0^t ds\ \hat{h}(s)\phi(s)}\langle f(\boldsymbol{\sigma}; \hat{\boldsymbol{h}})\rangle|_{\{h(\boldsymbol{\sigma},\boldsymbol{\phi})\}}$$

Note that

$$G(s,s') = -i\langle\langle\sigma(s)\hat{h}(s')\rangle\rangle = \frac{\partial}{\partial m(s')}\langle\langle\sigma(s)\rangle\rangle$$

so in fact we only ever need to evaluate averages involving spin variables only, allowing us to integrate out the conjugate fields altogether:

$$\langle\langle f(\boldsymbol{\sigma})\rangle\rangle \sim \int\{dh\}\{d\phi\}\ e^{-\frac{1}{2}\alpha\int_0^t dsds'\ \hat{h}(s)\left[(\boldsymbol{G}-\mathbf{1})^{-1}\boldsymbol{C}(\boldsymbol{G}^\dagger-\mathbf{1})^{-1}\right](s,s')\hat{h}(s')+i\sqrt{\alpha}\int_0^t ds\ \hat{h}(s)\phi(s)}\langle f(\boldsymbol{\sigma}; \hat{\boldsymbol{h}})\rangle|_{\{h(\boldsymbol{\sigma},\boldsymbol{\phi})\}}$$

$$\sim \int\{d\phi\}\ e^{-\frac{1}{2}\int_0^t dsds'\phi(s)\left[(\mathbf{1}-\boldsymbol{G})^{-1}\boldsymbol{C}(\mathbf{1}-\boldsymbol{G}^\dagger)^{-1}\right]^{-1}(s,s')\phi(s')}\langle f(\boldsymbol{\sigma}; \hat{\boldsymbol{h}})\rangle|_{\{h(\boldsymbol{\sigma},\boldsymbol{\phi})\}}$$

The variables $\phi(s)$ acquire a Gaussian distribution, characterised by $\langle\phi(s)\rangle = 0$ and

$$\langle\phi(t)\phi(s)\rangle = \left[(\mathbf{1}-\boldsymbol{G})^{-1}\boldsymbol{C}(\mathbf{1}-\boldsymbol{G}^\dagger)^{-1}\right](t,s) \tag{45}$$

Thus we arrive at

$$m(s) = \langle\langle\sigma(s)\rangle\rangle \tag{46}$$

$$C(s,s') = \langle\langle\sigma(s)\sigma(s')\rangle\rangle \tag{47}$$

$$G(s,s') = \frac{\partial}{\partial m(s')}\langle\langle\sigma(s)\rangle\rangle \tag{48}$$

For each realisation of the disorder noise variables $\{\phi(t)\}$ our equations describe an independent single-site process, in the form of a stochastic local field alignment, with the local fields (44). The brackets $\langle\langle\ldots\rangle\rangle$ denote averaging over this stochastic process, followed by averaging over the possible realisations of the disorder noise variables.

$$\frac{d}{dt}\hat{p}_t(\sigma) = \frac{1}{2}\left[1+\sigma\tanh[\beta h(\boldsymbol{\sigma},\boldsymbol{\phi};t)]\right]\hat{p}_t(-\sigma) - \frac{1}{2}\left[1-\sigma\tanh[\beta h(\boldsymbol{\sigma},\boldsymbol{\phi};t)]\right]\hat{p}_t(\sigma) \tag{49}$$

following

$$\hat{p}_0(\sigma) = \frac{1}{2}[1+m(0)]\delta_{\sigma,1} + \frac{1}{2}[1-m(0)]\delta_{\sigma,-1}$$

The corresponding result upon eliminating self-interactions, $J_{ii} \to 0$, is obtained by subtraction of the relevant term from the single-site effective alignment field (44): $h(\boldsymbol{\sigma},\boldsymbol{\phi};s) \to h(\boldsymbol{\sigma},\boldsymbol{\phi};s) - \alpha\sigma(s)$, equivalent with replacing (44) by

$$h(\boldsymbol{\sigma}, \boldsymbol{\phi}; s) = m(s) + \sqrt{\alpha}\phi(s) + \alpha\int_0^s ds'\left[(\mathbf{1}-\boldsymbol{G})^{-1}\boldsymbol{G}\right](s,s')\sigma(s') \tag{50}$$

## 2.3 Equilibrium Solutions and AT instability

Finally we show for sequential dynamics how, in the detailed balance case (50), the RS saddle point equations obtained within the framework of replica theory, as well as the AT instability, can be derived from the present dynamical formalism upon making suitable ansätze. It is not at all trivial that this can be done, in view of the non-Markovian nature of the single-spin problem (46-50). Indeed, the equations (46-50) are not an ideal starting point for constructing the equilibrium solution (although it must be possible). We will go back to an earlier stage: equations (38-40). Important tools will be the following two FDT relations:

$$G_{ij}(\tau) = -\beta\theta(\tau)\frac{d}{d\tau}C_{ij}(\tau) \tag{51}$$

$$s < s' < t: \qquad \frac{\partial^2\langle\sigma_(t)\rangle}{\partial\theta_j(s)\partial\theta_k(s')} = \beta\frac{\partial}{\partial\theta_k(s')}\frac{\partial}{\partial s}C_{ij}(t-s) \tag{52}$$

*Recovering the Replica-Symmetric Saddle-Point Equations.* Since in equilibrium initial conditions are required to be irrelevant, we shift the initial time $t_{\rm min} = 0$ to $t_{\rm min} = -\infty$ and the final time $t_{\rm max}$ to $t_{\rm max} = \infty$. Anticipating some necessary manipulations we restore the (time-dependent) external fields, which we choose to be site-independent: $\theta_i(t) = \vartheta(t)$ (equivalent to $\theta_i(t) = \xi_i^1\vartheta(t)$ in our original gauge). According to (39), which acquires an extra term $\boldsymbol{G}$ in the exponent due to the absence of self-interactions (detailed balance) and where we have made the normalisation explicit, one can now write spin-averages as

$$\langle\!\langle f\{\sigma\}\rangle\!\rangle = \int\frac{\{dh\}\{d\hat{h}\}}{\mathcal{N}(R,S)}e^{-\frac{1}{2}\alpha\int dsds'\hat{h}(s)R(s,s')\hat{h}(s')+i\int ds\ \hat{h}(s)[h(s)-m(s)-\vartheta(s)]}\langle f\{\sigma\}e^{-i\alpha\int dsds'\hat{h}(s)S(s,s')\sigma(s')}\rangle_{\{h\}} \tag{53}$$

with the two operators

$$R(s,s') = \left[(\mathbf{I}-\boldsymbol{G})^{-1}\boldsymbol{C}(\mathbf{I}-\boldsymbol{G}^\dagger)^{-1}\right](s,s') \qquad S(s,s') = \left[\boldsymbol{G}(\mathbf{I}-\boldsymbol{G})^{-1}\right](s,s') \tag{54}$$

(note: for the SK spin-glass one would have found $R(s,s') = C(s,s')$ and $S(s,s') = G(s,s')$), and where $\langle\Gamma(\boldsymbol{\sigma})\rangle_{\{h\}}$ denotes the average over the single-spin stochastic process process with time-dependent alignment fields $\{h(t)\}$:

$$\frac{d}{dt}\hat{p}_t(\sigma) = \frac{1}{2}\left[1+\sigma\tanh[\beta h(t)]\right]\hat{p}_t(-\sigma) - \frac{1}{2}\left[1-\sigma\tanh[\beta h(t)]\right]\hat{p}_t(\sigma)$$

We now make an equilibrium ansatz, where the correlation- and response-function are time-translation invariant, and where they obey the fluctuation dissipation theorem:

$$m(s) = m \qquad C(s,s') = C(s-s') \qquad G(s,s') = G(s-s') \tag{55}$$

$$G(\tau) = -\beta\theta(\tau)\frac{d}{d\tau}C(\tau) \tag{56}$$

with $C(\tau) = C(-\tau)$. Integration of the FDT gives $\int d\tau\ G(\tau) = \beta(1-q)$, or more generally:

$$\int d\tau\ G^n(\tau) = \int d\tau_1\ldots d\tau_n\ G(\tau_1-\tau_2)G(\tau_2-\tau_3)\ldots G(\tau_{n-1}-\tau_n)G(\tau_n) = [\beta(1-q)]^n \tag{57}$$

Time translation invariance of $C$ and $G$ implies the same for the above operators $R$ and $S$. In addition it ensures that all relevant operators will commute (since they can be simultaneously diagonalised on the Fourier basis). We now separate the persistent part of the correlation function (from (56) it follows that the persistent part of the response function is zero); this induces a similar separation for the operators $R$ and $S$:

$$C(\tau)=q+\tilde{C}(\tau)\qquad G(\tau)=\tilde{G}(\tau)\qquad \lim_{\tau\to\infty}\tilde{C}(\tau)=\lim_{\tau\to\infty}\tilde{G}(\tau)=0$$

$$R(\tau)=r+\tilde{R}(\tau)\qquad S(\tau)=\tilde{S}(\tau)\qquad \lim_{\tau\to\infty}\tilde{R}(\tau)=\lim_{\tau\to\infty}\tilde{S}(\tau)=0$$

in which (using (57)):

$$r=q\int d\tau d\tau'\left[(\mathbf{I}-\boldsymbol{G})^{-1}(t-\tau)(\mathbf{I}-\boldsymbol{G}^{\dagger})^{-1}\right](\tau')=\frac{q}{[1-\beta(1-q)]^2}\tag{58}$$

$$\tilde{R}(\tau)=\left[(\mathbf{I}-\tilde{\boldsymbol{G}})^{-1}\tilde{\boldsymbol{C}}(\mathbf{I}-\tilde{\boldsymbol{G}}^{\dagger})^{-1}\right](\tau)\qquad \tilde{S}(\tau)=\left[\tilde{\boldsymbol{G}}(\mathbf{I}-\tilde{\boldsymbol{G}})^{-1}\right](\tau)\tag{59}$$

In the averages (53) we can linearise the term generated by the persistent part $r$ of the operator $R$ with a Gaussian integral, and subsequently simplify the exponent by shifting the integration variables $\{h(s)\}$:

$$\langle\!\langle f\{\sigma\}\rangle\!\rangle=\int\frac{\{dh\}\{d\hat{h}\}}{\mathcal{N}(r,\tilde{R},\tilde{S})}e^{-\frac{1}{2}\alpha r\int dsds'\hat{h}(s)\hat{h}(s')-\frac{1}{2}\alpha\int dsds'\hat{h}(s)\tilde{R}(s-s')\hat{h}(s')+i\int ds\;\hat{h}(s)[h(s)-m-\vartheta(s)]}$$
$$\times\langle f\{\sigma\}e^{-i\alpha\int dsds'\hat{h}(s)\tilde{S}(s-s')\sigma(s')}\rangle_{\{h\}}$$

$$=\int Dz\int\frac{\{dh\}\{d\hat{h}\}}{\mathcal{N}(r,\tilde{R},\tilde{S})}e^{-\frac{1}{2}\alpha\int dsds'\hat{h}(s)\tilde{R}(s-s')\hat{h}(s')+i\int ds\;\hat{h}(s)\left[h(s)-m-z\sqrt{\alpha r}-\vartheta(s)\right]}\langle f\{\sigma\}e^{-i\alpha\int dsds'\hat{h}(s)\tilde{S}(s-s')\sigma(s')}\rangle_{\{h\}}$$

$$=\int Dz\int\frac{\{d\tilde{h}\}\{d\hat{h}\}}{\mathcal{N}(r,\tilde{R},\tilde{S})}e^{-\frac{1}{2}\alpha\int dsds'\hat{h}(s)\tilde{R}(s-s')\hat{h}(s')+i\int ds\;\hat{h}(s)\tilde{h}(s)}\langle f\{\sigma\}e^{-i\alpha\int dsds'\hat{h}(s)\tilde{S}(s-s')\sigma(s')}\rangle_{\{m+z\sqrt{\alpha r}+\vartheta+\tilde{h}\}}\tag{60}$$

with $Dz=(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}z^2}dz$.

Our final task is to show that for the observables $f\{\sigma\}=\sigma(t)$ and $f\{\sigma\}=\sigma(t)\sigma(t')$ (with $t\gg t'$) and in the equilibrium limit, with zero perturbation fields $\{\vartheta(t)\}$, the right-hand side of (60) does not depend on the details of the short-time parts $\tilde{R}$ and $\tilde{S}$. To demonstrate this we first prove that from the FDT for $C(\tau)$ and $G(\tau)$, see (56), it follows that the operators $\tilde{R}$ and $\tilde{S}$, given by (59), obey an identical FDT:

$$\tilde{S}(\tau)=-\beta\theta(\tau)\frac{d}{d\tau}\tilde{R}(\tau)\tag{61}$$

For $\tau<0$ this identity trivially holds, since $\tilde{G}(\tau<0)=0$. For $\tau>0$ we obtain (using commutation of the operators and the symmetry $\tilde{C}(\tau)=\tilde{C}(-\tau)$):

$$\beta\frac{d}{d\tau}\tilde{R}(\tau)=\beta\int_{-\infty}^{\tau}ds\;\frac{d}{d\tau}\tilde{C}(\tau-s)\left[(\mathbf{I}-\tilde{\boldsymbol{G}})(\mathbf{I}-\tilde{\boldsymbol{G}}^{\dagger})\right]^{-1}(s)+\beta\int_{\tau}^{\infty}ds\;\frac{d}{d\tau}\tilde{C}(s-\tau)\left[(\mathbf{I}-\tilde{\boldsymbol{G}})(\mathbf{I}-\tilde{\boldsymbol{G}}^{\dagger})\right]^{-1}(s)$$
$$=-\int_{-\infty}^{\tau}ds\;\tilde{G}(\tau-s)\left[(\mathbf{I}-\tilde{\boldsymbol{G}})(\mathbf{I}-\tilde{\boldsymbol{G}}^{\dagger})\right]^{-1}(s)+\int_{\tau}^{\infty}ds\;\tilde{G}(s-\tau)\left[(\mathbf{I}-\tilde{\boldsymbol{G}})(\mathbf{I}-\tilde{\boldsymbol{G}}^{\dagger})\right]^{-1}(s)$$
$$=\left[\frac{\tilde{\boldsymbol{G}}^{\dagger}-\tilde{\boldsymbol{G}}}{(\mathbf{I}-\tilde{\boldsymbol{G}})(\mathbf{I}-\tilde{\boldsymbol{G}}^{\dagger})}\right](\tau)$$

As a result we find:

$$\tau > 0: \qquad \tilde{S}(\tau) + \beta\frac{d}{d\tau}\tilde{R}(\tau) = \left[\tilde{\boldsymbol{G}}(\mathbf{I}-\tilde{\boldsymbol{G}})^{-1}\right](\tau) + \left[\frac{\tilde{\boldsymbol{G}}^{\dagger}-\tilde{\boldsymbol{G}}}{(\mathbf{I}-\tilde{\boldsymbol{G}})(\mathbf{I}-\tilde{\boldsymbol{G}}^{\dagger})}\right](\tau)$$

$$= \left[\frac{\tilde{\boldsymbol{G}}^{\dagger}}{\mathbf{I}-\tilde{\boldsymbol{G}}^{\dagger}}\right](\tau) = \left[\tilde{\boldsymbol{G}}+\tilde{\boldsymbol{G}}^{2}+\tilde{\boldsymbol{G}}^{3}+\ldots\right](-\tau) = 0$$

This completes the proof that in the limit of zero perturbation fields the operators $\tilde{R}$ and $\tilde{S}$ obey the FDT (61).

We are now in a position to analyse the effect on the average (60), considered as a functional of the operators $\tilde{R}$ and $\tilde{S}$, of varying these operators. Since at the end of our calculations we will put the perturbations $\{\vartheta(t)\}$ to zero, we need only consider variations which preserve the equilibrium relations $\tilde{R}(t) = \tilde{R}(-t)$ (which follows from (59)), $\tilde{S}(\tau)+\beta\theta(\tau)\frac{d}{d\tau}\tilde{R}(\tau) = 0$ and $\lim_{t\to\infty}\tilde{R}(t) = 0$. Partial integration over the fields $\{\hat{h}(s)\}$ allows us to convert all occurences of conjugate fields into partial derivatives with respect to perturbations, resulting in the effective replacement $\hat{h}(s) \to i\frac{\partial}{\partial\vartheta(s)}$:

$$\delta\langle\langle f\{\sigma\}\rangle\rangle = \int dt\left\{\delta\tilde{R}(t)\frac{\delta}{\delta\tilde{R}(t)} + \delta\tilde{S}(t)\frac{\delta}{\delta\tilde{S}(t)}\right\}\langle\langle f\{\sigma\}\rangle\rangle$$

$$= -\langle\langle f\{\sigma\}\rangle\rangle\frac{\delta\mathcal{N}}{\mathcal{N}} - \frac{1}{2}\alpha\int dt\left\{\delta\tilde{R}(t)\int ds\ \langle\langle\hat{h}(s)\hat{h}(s-t)f\{\sigma\}\rangle\rangle + 2i\delta\tilde{S}(t)\int ds\ \langle\langle\hat{h}(s)\sigma(s-t)f\{\sigma\}\rangle\rangle\right\}$$

$$= -\langle\langle f\{\sigma\}\rangle\rangle\frac{\delta\mathcal{N}}{\mathcal{N}} + \frac{1}{2}\alpha\int dtds\left\{\delta\tilde{R}(t)\frac{\partial^2\langle\langle f\{\sigma\}\rangle\rangle}{\partial\vartheta(s)\partial\vartheta(s-t)} - 2\beta\theta(t)\left[\frac{d}{dt}\delta\tilde{R}(t)\right]\frac{\partial\langle\langle\sigma(s-t)f\{\sigma\}\rangle\rangle}{\partial\vartheta(s)}\right\}$$

Due to causality, application of this result to the observable $f\{\sigma\} = 1$ gives $\delta\mathcal{N} = 0$, which we subsequently use to simplify the expression for the variation of $\langle\langle\sigma(\tau)\rangle\rangle$:

$$\delta\langle\langle\sigma(\tau)\rangle\rangle = \frac{1}{2}\alpha\int dtds\left\{\delta\tilde{R}(t)\frac{\partial^2\langle\langle\sigma(\tau)\rangle\rangle}{\partial\vartheta(s)\partial\vartheta(s-t)} - 2\beta\theta(t)\left[\frac{d}{dt}\delta\tilde{R}(t)\right]\frac{\partial\langle\langle\sigma(s-t)\sigma(\tau)\rangle\rangle}{\partial\vartheta(s)}\right\}$$

$$= \frac{1}{2}\alpha\int dt\ \delta\tilde{R}(t)\int ds\left\{\frac{\partial^2\langle\langle\sigma(\tau)\rangle\rangle}{\partial\vartheta(s)\partial\vartheta(s-t)} + 2\beta\frac{\partial}{\partial t}\left[\theta(t)\frac{\partial\langle\langle\sigma(s-t)\sigma(\tau)\rangle\rangle}{\partial\vartheta(s)}\right]\right\}$$

$$= \frac{1}{2}\alpha\int_0^\infty dt\ \delta\tilde{R}(t)\int ds\left\{\frac{\partial^2\langle\langle\sigma(\tau)\rangle\rangle}{\partial\vartheta(s)\partial\vartheta(s-t)} + \frac{\partial^2\langle\langle\sigma(\tau)\rangle\rangle}{\partial\vartheta(s)\partial\vartheta(s+t)} + 2\beta\frac{\partial}{\partial t}\frac{\partial\langle\langle\sigma(s-t)\sigma(\tau)\rangle\rangle}{\partial\vartheta(s)}\right\}$$

$$= \alpha\int_0^\infty dt\ \delta\tilde{R}(t)\int ds\left\{\frac{\partial^2\langle\langle\sigma(\tau)\rangle\rangle}{\partial\vartheta(s)\partial\vartheta(s-t)} + \beta\frac{\partial}{\partial t}\frac{\partial\langle\langle\sigma(s-t)\sigma(\tau)\rangle\rangle}{\partial\vartheta(s)}\right\} \tag{62}$$

At these stage we evoke the FDT relation (52), which in the present case and for $i = j = k$ leads to

$$t > 0: \qquad \frac{\partial^2\langle\langle\sigma(\tau)\rangle\rangle}{\partial\vartheta(s)\partial\vartheta(s-t)} = \beta\theta(\tau-s)\frac{\partial}{\partial(s-t)}\frac{\partial\langle\langle\sigma(s-t)\sigma(\tau)\rangle\rangle}{\partial\vartheta(s)}$$

with which we obtain the desired equilibrium result

$$\lim_{\vartheta\to 0}\delta\langle\langle\sigma(\tau)\rangle\rangle = \alpha\beta\int_0^\infty dt\ \delta\tilde{R}(t)\int_{-\infty}^{\tau} ds\left\{\frac{\partial}{\partial t} - \frac{\partial}{\partial t}\right\}\frac{\partial\langle\langle\sigma(s-t)\sigma(\tau)\rangle\rangle}{\partial\vartheta(s)} = 0$$

We conclude that we can calculate the equilibrium value $\lim_{\vartheta\to 0}\langle\langle\sigma(\tau)\rangle\rangle$ by choosing *any* pair $\{\tilde{R},\tilde{S}\}$ that obey the FDT (61). In particular, we are allowed to make the trivial choice $\tilde{R}=\tilde{S}=0$ in applying (60) to the observable $f\{\sigma\}=\sigma(\tau)$, and find the equilibrium result

$$\lim_{\vartheta\to 0}\langle\langle\sigma(\tau)\rangle\rangle=\int Dz\;\tanh\beta[m+z\sqrt{\alpha r}] \tag{63}$$

In principle we could do a similar exercise to show that for $\tau\gg\tau'$ the equilibrium average $\lim_{\vartheta\to 0}\langle\langle\sigma(\tau)\sigma(\tau')\rangle\rangle$ (which will produce $q$) does not depend on the details of the short-time parts $\tilde{R}$ and $\tilde{S}$ either. This would involve higher order FDT's. However, there is a quicker way to obtain this result. From (60) we infer

$$\int dt'\tilde{G}(t,t')=\int dt'\frac{\partial\langle\langle\sigma(t)\rangle\rangle}{\partial\vartheta(t')}=-i\langle\langle\Big[\int dt'\hat{h}(t')\Big]\,\sigma(t)\rangle\rangle=\frac{\partial}{\partial m}\langle\langle\sigma(t)\rangle\rangle$$

By taking the (equilibrium) limit of zero perturbation fields, and upon using the equilibrium relation $\int d\tau\;\tilde{G}(\tau)=\beta(1-q)$ and our result (63), we find

$$\beta(1-q)=\beta\int Dz\left[1-\tanh^2\beta[m+z\sqrt{\alpha r}]\right]$$

The combination of this equation, the saddle-point equation $m=\lim_{\vartheta\to 0}\langle\langle\sigma(\tau)\rangle\rangle$, and the definition (58), i.e.

$$m=\int Dz\;\tanh\beta[m+z\sqrt{\alpha r}]\qquad q=\int Dz\;\tanh^2\beta[m+z\sqrt{\alpha r}]\qquad r=\frac{q}{[1-\beta(1-q)]^2}$$

indeed give the saddle-point equations obtained with the replica formalism, within the replica-symmetric ansatz. In order to arrive at this result we had to assume equilibrium (in the form of various FDT relations) and the absence of anomalous response. This is in nice agreement with our interpretation of the meaning of replica symmetry. Although these equations suggests a relatively simple system with a Gaussian distribution of local alignment fields, it is important to realise this is very misleading. Due to the detailed balance property the short-time parts $\tilde{R}$ and $\tilde{S}$ may not play a role in our calculation of $m$ and $q$, but they are certainly non-zero (which already follows from (50)), and the local field distribution, if calculated within equilibrium statistical mechanics, does come out to be non-Gaussian, even in equilibrium.

Finally, it is possible to show that the AT instability corresponds in the present dynamical formalism to the condition tof anomalous response, i.e. where the characteristic time for the relaxation of the non-persistent contributions to the various operators ($\tilde{C}$, $\tilde{G}$, $\tilde{R}$, $\tilde{S}$) diverges. This also results in a violation of the FDT relations, e.g. via a divergence of the series expansion in powers of $\tilde{\boldsymbol{G}}$, used to prove the FDT for the pair $\{\tilde{R},\tilde{S}\}$. I will not derive such results here.